THE

MAGNETIC CIRCUIT

THE MAGNETIC CIRCUIT

IN THEORY AND PRACTICE

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THE AUTHOR'S PREFACE

THE plan of the present work arose out of a lecture on the Magnetic Circuit and its Measurement, which I delivered on the occasion of the International Congress of Electricians at Frankfort in September 1891. From several sides the desire was expressed for a systematic and critical account, from the physical point of view, of the more important developments in this direction. Hitherto such an account had been wanting, and this deficiency I have endeavoured to supply, however imperfectly.

The important action which the rapid rise of electro-technology in the last decade has exerted on those branches of physics which form its basis, has been abundantly insisted on, and almost universally recognised. Electro-technology seems at present to have entered on a phase of quieter development; and from the scientific point of view the time appears suitable to survey the position, critically to investigate results of very unequal value, often hastily brought to light amidst the bustle of practical work, and to blend the older as well as the more recent results into one consistent exposition.

I have in general directed my efforts chiefly to describing the actual state of theoretical and experimental inquiry. Any attempt to chronicle completely the manifold phases of its previous development seemed the more unnecessary since a new edition of Wiedemann's 'Lehrbuch der Elektricität' is now appearing, in the third volume of which the subject in question is historically treated in an exhaustive manner.

I have made an exception in the seventh chapter, in which I have endeavoured to give the history of the analogy between the magnetic circuit and various other kinds of circuits, an analogy which, though long known, has been specially dwelt upon in recent times.

The book forms two parts. The two introductory chapters are intentionally concise, and yet as elementary as possible, so as to enable anyone not completely familiar with mathematical physics to understand the later parts of the book. I have made no attempt to state completely the results of investigations into ferromagnetic induction, which of late have been so much pursued, since a full account of them has recently appeared from the authoritative pen of Professor Ewing. Hence the knowledge of these results is assumed, and the theory of the magnetic circuit based on them, as well as on earlier theoretical investigations. The fundamental type of these circuits is the radially divided toroid introduced at the outset, which is discussed in detail in Chapter V., and continually referred to afterwards. The explanation of ferromagnetism by pre-existing elementary magnets capable of being directed, and the comparison of these magnets to rotatory processes, whether vortices (Lord Kelvin), molecular currents (Ampère) rotating electrical particles (Weber), or ionic charges (Richarz), could only be treated very superficially.

In Chapters III. and IV. the outlines of the theory of 'rigid' magnets on the one hand, of absolutely 'soft' cores on the other, are briefly summarised. The mode of treatment is similar to that of Maxwell, which had been followed among others by Professor Chrystal in his article on Magnetism in the 'Encyclopædia Britannica,' and by MM. Mascart and Joubert in their Treatise. From the nature of the case this treatment could not be elementary. But by adopting a geometrical or graphical mode of representation, as well as by avoiding purely analytical refinements, I have endeavoured to attain that clearness and distinctness to the want of which it may be due that in many cases

ignorance or doubt exists as to the long-established theory in question. I have made no use of actual quaternion methods in their original form or in the modified one advocated by Mr. Heaviside, as their knowledge can scarcely be considered sufficiently widespread.

Unlike the purely scientific method previously employed, the second part of the book is treated more from the point of view of applied physics. In Chapter VI. the general properties of the magnetic circuit are discussed. Chapter VII., in which historical treatment preponderates, has been already mentioned. Chapters VIII. and IX. treat briefly the application of the principles developed to the principal machines and apparatus used in actual practice or in the laboratory. Such a concise account of the chief applications of the science may be welcome to many physicists on the one hand, and on the other may interest practical electricians, as giving the scientific foundation of their special subject.

The last two chapters, finally, are devoted to experimental methods of measurement. Wherever the more important results of allied branches of mathematical or experimental physics are assumed to be known, I have referred to the original passages in the text-books mentioned above or in others. Numerous references facilitate, moreover, the study of details in original papers.

The chief contents of §§ 81, 94, 95, 109, 124, 139, 154, 158, 179 have hitherto, so far as I know, either not been published at all or only briefly and without proof. In other places I have given new proofs, wherever necessary, but have not always mentioned them. I have throughout endeavoured to adopt a suitable nomenclature, which has not always been an easy matter, considering the confusion which prevails in this respect. The nomenclature and notation of all the more important conceptions are logically retained throughout, and are given at the end in the form of a tabular summary.

To Professor EWING, Lord KELVIN, the late Professor KUNDT, Dr. H. LEHMANN, Dr. LINDECK, Dr. NAGAOKA, Professor PLANCK, Dr. RAPS and Dr. RUBENS, who have each been good enough to look through portions of the proof sheets, I am indebted for much valuable advice, and in conclusion I wish to express my best thanks to them.

H. DU BOIS.

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TRANSLATOR'S PREFACE

THE Preface by the Author explains so completely the scope of the present work, that no addition in this respect is required from me.

In offering the translation I have to acknowledge the valuable assistance of Dr. C. V. BURTON, of University College, London, who translated a considerable portion, and revised the rest of the more purely mathematical part of the work: to Dr. E. H. BARTON, of University College, Nottingham, I am also indebted for having looked through and revised the more purely technical portions.

I must further express my acknowledgments to Dr. E. TAYLOR JONES, now lecturer in University College, Bangor, who was studying under Dr. du Bois while some of these sheets were passing through the press, and who kindly read them.

I am still more indebted to Dr. du Bois himself, who, an excellent English scholar, read the proofs and added to the translation a series of notes embodying the latest results of scientific investigation in magnetism.

E. ATKINSON.

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THE MAGNETIC CIRCUIT

PART I

THEORY

CHAPTER I

INTRODUCTION

§ 1. **The Electromagnetic Field.**—'The electric current, or, more generally, electricity in motion, is the only known source of any kind of magnetism, and more particularly also of terrestrial magnetism,' as may, with great probability, be assumed. 'Magnetic iron ore and other bodies occurring in nature in the magnetic condition manifestly owe their magnetism to that of the earth, or, in some cases, no doubt, to the direct action of electrical discharges.'[1]

We start, therefore, by taking the fact as known, that a conductor along which a current is passing produces in its vicinity a peculiar condition which is called an *electromagnetic*, or, more briefly, a *magnetic field*. The air which in the ordinary conditions of experiment occupies this space plays only a very subordinate part, to which we shall afterwards refer (§ 7). In the phenomena to be subsequently described we shall assume that they take place in a vacuum.

The condition in question manifests itself among other things by the fact that, on the one hand, forces are exerted in the magnetic field on other conductors carrying currents; and, on the other, that momentary currents are induced in conductors when, and only in so far as, the condition in question is altered, either as regards position or value, and particularly when it either suddenly appears or completely vanishes. Movable conductors conveying a current are therefore put in motion. On the other hand, momentary currents are induced in movable con-

[1] Compare W. von Siemens, *Wied. Ann.* vol. **24**, p. 94, 1885.

ductors whenever the magnetic condition with reference to them is altered by their motion. This must suffice for the general characterisation of the phenomena in question, the experimental details of which must be assumed to be known.

Both these classes of phenomena, the *electrodynamic* and the *inductive*, are equally well suited theoretically for a complete determination of the magnetic condition. A whole series of methods for practically attaining this object has been developed, which we shall consider more closely further on (Chapter X.).

§ 2. **The Magnetic Condition as a Directed Quantity.**—For our present object, which is mainly theoretical, the following elementary arrangement will be sufficient. Let a metal wire be bent so as to form a plane loop, which encloses the area S; let what is called the *secondary* circuit, of which it forms a part, have the resistance R. Let the momentary current induced in the wire cause a quantity of electricity, Q, to be displaced, the absolute value of which can be measured by any suitable arrangement. By means of such a movable coil, which is an '*exploring coil*,' we can investigate the magnetic field, and, as it were, make a topographical survey [1] of it.

We have in the first place to investigate what takes place if we leave the small exploring coil in one place, and only alter its direction. This is defined by the direction of the perpendicular on one side to the plane of the coil. If we let the perpendicular sweep through all possible directions in space, we find that there are two positions, and these two exactly opposite each other, in which a maximum quantity of electricity is induced, when the current is made or broken, in what is called the *primary* conductor.

In all other directions of the perpendicular to the coil smaller quantities of electricity are obtained; they are, in fact, in each case proportional to the cosine of the inclination to the direction of maximum induction. It follows from this that for all directions of the perpendicular which lie in the plane at right angles to that special direction the quantity of electricity

[1] In practice a *ballistic galvanometer* is almost always used for such experiments; this itself depends indirectly on actions similar to those here described. Compare Faraday, *Exp. Researches*, vol. 3, p. 328.

induced is zero—that is, no induction takes place. All this tends to indicate that we are here dealing with one of those physical conditions which can only be completely defined by a vector. We must first explain more in detail the important idea expressed by this word, before proceeding to further experiments with the exploring coil.

§ 3. **Elementary Conceptions of Quaternions.**—Although we shall in the sequel make no use of special quaternion methods, the most elementary extremely useful conceptions and notations of that branch of science will be frequently applied.[1]

Physical quantities may be divided into two groups, those of the directed and those of the undirected, which are distinguished as *vectors* and *scalars*. With regard to scalars nothing need be said; the general properties of physical quantities, of their numerical values, as well as of their units, are supposed to be known. But vectors, from the very fact of their being directed, have, in addition, special properties; we are here chiefly concerned with the law of their geometrical addition.

The sum of two or more vectors is, in general, not equal to the sum of their numerical values. It is obtained in a manner which is generally known, by the way in which a vector quantity of frequent occurrence, force, is geometrically added; that is, by the construction of a parallelogram for two, and of a polygon[2] for several vectors. In accordance with this, a vector may, conversely, be resolved into any number of components having given directions, in particular those of the axes of coordinates. The numerical value of a vector component is obtained by multiplying that of the vector itself into the cosine of the angle between the two directions.

On account of the essential differences between the mathematical operations to be performed with scalars and vectors, it is desirable to be able to see from the symbol for a quantity to which of the two groups it belongs. Hence it is usual to denote, by German capitals, those quantities the vector character of which

[1] For further details reference must be made to the important works of Grassman, of Hamilton, and of Tait. See also O. Heaviside, *Electromagnetic Theory*, London, 1893.

[2] That is in the general case of a broken polygon of straight lines in space.

is to be clearly shown.[1] We shall adopt this plan, and refer to Chapter III. for further geometrical considerations as to vectors.

§ 4. **Magnetic Intensity.**—After this unavoidable digression we return to the magnetic vector. We now attempt to determine its numerical value, by placing the exploring coil in the direction of maximum induction, and investigating on what variable quantities the quantity Q of induced electricity depends. We shall then find that it is proportional to the area S of the winding, and inversely proportional to the resistance R. These factors, which have obviously no connection with the magnetic condition, we eliminate by forming the expression QR/S; we have then to consider this as the absolute measure for the magnetic condition, and accordingly put

$$(1) \qquad . \quad . \quad . \quad . \quad \mathfrak{H} = \frac{QR}{S}$$

It is here to be observed that if Q, R, and S are expressed in any consistent system of measurement, the expression QR/S measures the magnetic condition also in this system. In such considerations as we are here concerned with, it is, in fact, the practice to adopt exclusively the electromagnetic C.G.S. system.[2] In accordance with this S in the above equation is expressed in square centimetres, Q in decacoulombs, R in millimicrohms.

The quantity $\mathfrak{H}$ thus defined absolutely, we shall call the *intensity* of the magnetic field; the symbol chosen denotes its vector character. Its direction is that of the perpendicular on one side to the coil in the position of maximum induction, and with the condition that the sense of the current, induced by the cessation of the field, stands in the same geometrical relation to the direction of the field, as the sense in which the hands of a clock move, to the direction from the dial towards the works.

[1] This practice was introduced by Maxwell in his *Treatise on Electricity and Magnetism.* Instead of this, we find in many English authors block letters in the middle of the text, which, however, are scarcely an ornament; the choice of a notation is, of course, a somewhat unimportant matter of taste.

[2] It is not necessary to enter here more minutely upon the theory of absolute systems of measurement, as there are several excellent special works on the subject.

If we imagine *lines of intensity*[1] in space—that is, curves whose tangent at each point gives the direction of the intensity—we have by these a means of making evident the distribution of the direction of the magnetic vector in space. We can in many cases, as we shall afterwards see, draw conclusions as to the numerical value of a given vector from the course of such groups of lines.

A method frequently used for graphically representing lines of intensity in two dimensions consists in using the finest dust from iron filings; when this is scattered on a sheet of stout paper, which is then gently tapped, the dust arranges itself in the direction of these lines, and can afterwards be fixed.

§ 5. **Magnetic Field of Straight Conductors.**—The more detailed geometrical investigation of the field which is produced by linear conductors of various shapes in the space near them we need not go into.[2] It will be sufficient to mention briefly a few special cases which we most frequently meet with.

Here again, as has been before stated, all the equations are to be interpreted in the electromagnetic C.G.S. system. Currents, for instance, are then to be measured in deca-ampères, and linear dimensions in centimetres. On this depends the simplicity of the equation, and the avoidance of arbitrary constants.

A. *Straight Element of Current.*—This, which is exclusively a mathematical abstraction, cannot physically be realised; it presents, however, considerable interest, for by integrating the corresponding elementary equation over closed conductors we obtain results which are susceptible of exact experimental confirmation. A straight infinitely short element of conductor of length dL, conveying a current, produces at a point at a distance

[1] In ordinary language we frequently speak of *magnetic force* and *lines of force*, the latter expression being also frequently used for those curves which we shall more logically introduce as lines of induction (§ 61). Maxwell himself has, however, given the preference to the word 'intensity,' as undoubtedly follows from the second edition of his *Treatise*—so far as the author himself revised it (see particularly **1**, § 12). E. Cohn, 'Systematik der Elektricität,' *Wied. Ann.* vol. **40**, p. 628, 1890, as well as Hertz, *Untersuchungen*, p. 30, Leipzig, 1892, agree with this.

[2] Compare Mascart and Joubert, *Electricité et Magnétisme*, vol. **1**, §§ 442–506, Paris, 1882.

r a magnetic intensity, the numerical value of which, $\delta \mathfrak{H}$, is given by the following equation:

$$(2) \quad . \quad . \quad . \quad \delta \mathfrak{H} = \frac{I \delta L \sin a}{r^2}$$

in which a is the angle between the direction of the element and the straight line which connects it with the point in question. The direction of the field runs at right angles to the meridianal plane, which can be drawn through this straight line and the element.

B. *Straight Conductors.*—Let part of a circuit consist of a long straight piece; the other parts may be at a greater distance. In the immediate neighbourhood of the straight piece its influence preponderates, and is exerted as follows: The magnetic field is at each point at right angles to the plane passing through it and the straight piece, and accordingly the lines of intensity are evidently concentric circles. The numerical value of the intensity is given by the equation

$$(3) \quad . \quad . \quad . \quad . \quad \mathfrak{H} = \frac{2 I}{r}$$

in which r is the distance of the given point from the conductor—that is, the radius of the corresponding circular line of intensity. This relation, in which the intensity of the field is inversely proportional to the distance from the conductor, is known as 'Biot-Savart's law,' these physicists having first discovered it by experiment.

§ 6. **Magnetic Field of a Circular Conductor.**—C. *Circular Conductor.*—A plane circular conductor produces in its axis a field in the direction of the axis. Let r be the radius of the circle, x the distance along the axis, measured from the centre where it cuts the plane of the circle, $z = \sqrt{x^2 + r^2}$ is the distance of a point on the axis from the circumference. The numerical value of the field-intensity at the distance x is

$$(4) \quad . \quad . \quad . \quad . \quad \mathfrak{H} = \frac{2 \pi I r^2}{z^3}$$

This expression has a maximum value at the centre, where $x = 0$; there we have then

$$(5) \quad . \quad . \quad . \quad . \quad \mathfrak{H} = \frac{2 \pi I}{r}$$

For points on the axis at such a distance that in comparison with it the linear dimensions of the coil may be neglected we may, in equation (4) put $z = x$, the distance from the centre of the coil. Further, πr^2 is the area of the circular conductor, that is the area of the coil S; this, moreover, need not necessarily be a circle, and with several windings the total area $\Sigma(S)$ enters into the equation, which may then be written

$$(6) \quad . \quad . \quad . \quad \mathfrak{H} = \frac{2 I \Sigma (S)}{x^3} = \frac{2 \mathfrak{M}}{x^3}$$

In this the expression $I \Sigma (S)$ is put $= \mathfrak{M}$, as this quantity determines the action at a distance of the windings. $\mathfrak{M}$ is called its *magnetic moment* (comp. § 22).

The action of a circular conductor at points outside the axis can, in general, be only expressed by means of spherical harmonics.

D. *Long Coil.*—Of special importance is the action at a point in the interior of a uniformly wound bobbin, the length of which is very great in comparison with its cross-section. If the *number of turns* is n, the length L, the magnetic intensity at all points sufficiently distant from the ends is

$$(7) \quad . \quad . \quad . \quad . \quad \mathfrak{H} = \frac{4 \pi n I}{L}$$

It is therefore determined geometrically by n/L—that is, by the number of windings per unit length—and depends neither on the shape nor on the area of the winding. At the ends themselves the value of $\mathfrak{H}$, as a little consideration will show, is only one-half the above,

$$(8) \quad . \quad . \quad . \quad . \quad \mathfrak{H} = \frac{2 \pi n I}{L}$$

and diminishes rapidly as the distance in the direction of the axis increases, until at greater distances equation (6) again holds. Equation (7) holds also for long *closed* coils—that is to say, those whose axis forms any closed curve the *centroid* of the coil, and accordingly has no ends; such coils we shall frequently have to take into consideration.

As regards the relation between the direction of the current in the conductors and the direction of the field pro-

duced, this, in cases A, B, C, D, is always the same as that between the direction of clock-motion and the direction from the dial to the works, and is independent of whether the electrical circuit is straight, and the line of magnetic intensity is curved or the reverse.

§ 7. **Diamagnetic and Paramagnetic Substances.**—We have, as has been already stated, always assumed that the medium in which the phenomena take place is a vacuum. But with the exception of certain mixtures specially prepared for this purpose, the magnetic behaviour of all material substances more or less differs from what has been described.[1] In order to investigate this, we introduce any given isotropic substance into a definite place in the magnetic field, and by preference we use it in the form of a sphere in order that, its shape being quite symmetrical, the exploring coil may fit in all directions.

By means of a momentary current induced in the latter, we can then, as above (§ 2), ascertain the magnetic condition of the sphere, and we shall find as follows. The magnetic condition in this case also is a vector, which has exactly the same direction as the intensity $\mathfrak{H}$ would have in a vacuum in the same places. With by far the greater number of substances, the numerical value of that vector, which as before is measured by the quantity of electricity induced per unit surface and unit of resistance, differs only in the fourth or fifth decimal place from the value obtained for a vacuum.[2] Two cases may here be distinguished.

In by far the greater number of substances the magnetic vector is very slightly *less* than the corresponding one in a vacuum; these belong to the group which Faraday called *diamagnetic*.

If, however, the vector in the substance has a somewhat *greater* value than in a vacuum, it is classed with what is called the *paramagnetic* group.

It is usual to say that the magnetic condition in the substance is *induced* by that which would obtain in the same place in a vacuum, and for the measure of which we have above introduced

[1] Faraday, on the magnetic condition of all matter (*Exp. Researches*, vol. 3, series 20 and 21, 1845).

[2] These small differences can in fact be only qualitatively shown by means of special experimental arrangements; quantitative measurements are out of the question, otherwise than by rather refined methods.

the magnetic intensity. There is no adequate reason for diverging from this mode of expression, but it must be expressly stated that the process in question must not at all be considered as being explained by assigning to it the term *magnetic induction.* In fact, so far as we know at present, the magnetic condition in all para- and diamagnetic substances at a given temperature depends exclusively on the intensity of the field in which they are placed, and its value is even proportional to that intensity.

§ 8. **Ferromagnetic Substances and Interferric.**—It is otherwise with a small number of substances which in this respect claim an exceptional position. The value of the magnetic condition induced in them is not proportional to that of the space which they occupy; in addition to this it depends on the manner in which the substances have acquired their condition at the time considered. Their 'magnetic history' exerts an influence on their behaviour which is principally affected by the periods nearest in time to that in question. The magnetic condition of such bodies *lags*, as it were, behind the magnetic intensity which induces it. Accordingly at present the whole of the phenomena which belong to this class are included in the name *hysteresis* (Greek *ὑστερέω*, to lag or remain behind).[1]

The phenomenon of magnetic retentiveness, which historically was the first observed, and which was formerly the starting point of all considerations, is only a special case of hysteresis, as the name denotes. Regarded from this point of view, it must be considered as a residue of the action of earlier causes, which may indeed for thousands of years have ceased to exist.

These substances, by the properties which have been mentioned, are apparently different from all others, although it has not hitherto been possible to give any valid reason for this exceptional behaviour. They are accordingly classed in a separate group, the *ferromagnetic.*[2] As, further, their magnetic properties are especially prominent, they have been observed since the most

[1] Our knowledge in this field is due principally to the researches of Warburg and of Ewing.

[2] By many authors the terms 'ferromagnetic' and 'paramagnetic' are used pretty indiscriminately. For the present it may be as well to keep the two groups separate. In what follows, we shall usually drop the prefix 'ferro-.' In doing so we follow the ordinary usage, which is not likely to lead to confusion, as we shall deal exclusively with ferromagnetic bodies.

remote times. Their more minute investigation is, however, an acquisition of the modern scientific era.

So far as is at present known, the group in question at ordinary temperatures consists of three metals which are also chemically allied—iron, cobalt, and nickel, together with some of their alloys, and compounds with one another or with other elements, as, for instance, carbon, oxygen, manganese, aluminium, mercury. A sharp delimitation of the ferromagnetic from the paramagnetic group may be made at present, but it is not improbable that in the future this will disappear.[1]

We shall deal in the sequel with combinations, consisting partly of ferromagnetic substance and partly of such as are not ferromagnetic. From the far more pronounced character of the ferromagnetic part the special nature of the latter does not at all come into play. This may either be paramagnetic or diamagnetic, and may be in any state of aggregation. In most cases in practice it will consist of the surrounding air, but may just as well be supposed to be filled with any other solid, liquid, or gas. We shall frequently call this the *interferric*,[2] and we may regard it with sufficient approximation as magnetically *indifferent* —that is, as not being different from vacuum.

As regards the ferromagnetic substance, it is tacitly assumed that it is homogeneous, isotropic, and free from any current (§ 54). When the contrary is not expressly mentioned, hysteretic properties will be disregarded; it may, for instance, be assumed that the ferromagnetic substance is exposed to shocks, or to superposed alternating fields, which act in opposition to hysteresis;[3] and it is more particularly assumed that after the cessation of the magnetising influences, no magnetic properties are left behind.

§ 9. **Magnetically Indifferent Toroid.**—We shall now pass to the case of a ring of material indifferent to magnetism, as represented in fig. 1. Let its section be circular and of area S,

[1] [This discontinuity appears to be already bridged over by the recent valuable researches of P. Curie, Thesis No. 840, Paris, 1895. H. d. B.]

[2] German *interferricum*, a word imitated from the French *entrefer* which was introduced by Hospitalier, and which has been frequently adopted on the Continent. When the interferric consists of air only, it is usually called the '*air-gap*'; the expression in the text is more general.

[3] Compare Gerosa and Finzi, *Rend. R. Ist. Lomb.* vol. **24**, p. 149, 1891; *Rend. R. Lincei* [8], vol. **7**, p. 253, 1891; *Wied. Beiblätter*, vol. **16**, p. 329, 1892.

and let the diameter $2r_1$ of the dotted circle, the *centroid*, be considerable in comparison with the thickness of the ring. Let such a body, which is briefly called a *toroid* or *anchor ring*, be uniformly covered with n_1, primary, as well as with n_2, secondary, windings. For the sake of simplicity, let the wire of both coils be assumed to be infinitely thin, so that the area of each separate winding is also S. Let us suppose, moreover, another coil constructed in exactly the same way, but with the essential difference that the core consists of a ferromagnetic instead of an indifferent substance.

Let now both the primary coils be inserted in series in an electric circuit, and each of the two secondaries in the circuit of any experimental arrangement, by means of which the momentary currents induced in them may be determined in absolute measure. In practice, a ballistic galvanometer is mostly used for this purpose. Let the resistance of these secondary circuits, coil, galvanometer, and leads be R. Let the current through the primary coils be now suddenly closed, and let us consider, in the first place, the behaviour of the magnetically indifferent toroid.

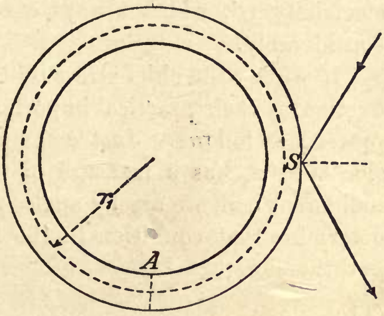

FIG. 1.

Within the closed coil which surrounds it, and of which the mean length is $2\pi r_1$, this current I will produce a magnetic field the intensity of which (§ 6, equation 7) is given by the expression

$$(9) \qquad \mathfrak{H} = \frac{4\pi n_1 I}{2\pi r} = \frac{2 n_1 I}{r_1}$$

It follows, further, from the principles of the induction of momentary currents by changes in magnetic condition, which we have already discussed, that the quantity of electricity Q passing through the ballistic galvanometer, and induced in the secondary coil, is given by the following equation (comp. equation 1)

$$(10) \qquad Q = \frac{n_2 \mathfrak{H} S}{R}$$

From this we see that the expression $Q R / n_2 S$, made up of quantities which can be directly observed, gives a direct measure for the magnetic intensity in the magnetically indifferent toroid. But from our former explanations, the latter measures the directed magnetic condition in the substance of that body. It is here once more to be insisted on that, in interpreting the equations, use is always to be made of the electromagnetic C.G.S. system.

§ 10. **Ferromagnetic Toroid; Magnetic Induction.**—Let us now turn to the ferromagnetic toroid; under the same circumstances, its secondary coil will be traversed by a quantity of electricity Q', which always exceeds Q, and, in general, very considerably.

If we lay the chief stress on the phenomena of magnetic induction, as their practical importance requires, we may obviously proceed as follows. Just as as we have considered the expression $Q R / n_2 S$ as a measure of the magnetic condition in the indifferent coil, we are by analogy entitled to use $Q' R / n_2 S$ to determine that condition in the ferromagnetic substance. We put, therefore,

$$(11) \quad . \quad . \quad . \quad . \quad \frac{Q' R}{n_2 S} = \mathfrak{B}$$

and call the value, thus defined, the *induction*. This name was introduced by Maxwell;[2] it recalls that the vector can be arrived at by considering one of the most important manifestations of varying magnetic condition: the induction of electromotive impulses in adjacent conductors.

Yet there are several such manifestations, and among them processes which occur not merely at the moment of change. And although induction has hitherto undoubtedly played the chief part, especially from the practical point of view, there is no physical reason for assigning to it a preferential position under

[1] Compare in this respect Chapter IV. § 64.

[2] Maxwell, *Treatise*, 2nd edition, vol. 2, § 400. This term, of such general use, has for a strictly definite idea the disadvantage of being applied in a perfectly indeterminate manner to a group of physical phenomena, not to speak of non-physical branches of knowledge. Mistakes are, however, not likely to occur, although two of these groups, as follows from the above, are in direct relation with the idea in question (compare § 113).

all circumstances. It is usual to assert in reference to such a ferromagnetic ring as the above that its magnetic condition can in no way be externally perceived. This is only correct if we are thinking of the apparent actions at a distance which we have been accustomed to observe as arising from bar magnets or bodies of similar shape.

For, in the first place, the perimeter of the toroid changes on magnetisation, though only to an insignificant extent.

If, secondly, a small portion of the surface is polished, so as to reflect, and a pencil of light is reflected from this, a change of the state of polarisation in the reflected light is observed on magnetisation.[1] In the particular case in which the light is polarised in the plane of incidence (that is, of the plane of the figure, fig. 1, p. 11), and the angle of incidence is about 60°, a simple rotation of the plane of polarisation is observed.

It might, thirdly, be added that on magnetisation, internal stresses (§ 101), peculiar (rotatory) properties of electric and thermal conductivity, as well as changes of thermo-electric behaviour, of specific heat, &c. are met with.

From this we see that the existence of the magnetic condition in the ferromagnetic substance is manifested in various ways, even when no action at a distance in the ordinary sense is to be observed. We shall even see that it is the absence of this in the case of a toroid which marks this case as typical; for an apparent action at a distance forms no criterion for the existence of a uniform magnetic condition; it proceeds, in fact, only from places where there is a local variation, or even a sudden cessation of such conditions, as will be more fully explained further on.

§ 11. **Saturation. Magnetisation.**—The phenomena last mentioned all show the peculiarity that with an unlimited increase of the magnetising field they do not increase in a corresponding manner. They undergo, on the contrary, continually smaller increments, until finally they become practically constant. The magnetic condition appears then, as it were, *saturated*, to use the ordinary expression.

The induction $\mathfrak{B}$, defined and deduced as above from magneto-electrical phenomena, never, on the contrary, attains

[1] This phenomenon was first observed by Kerr, *Phil. Mag.* [5], vol. **3**, p. 321, 1877, and vol. **5**, p. 161, 1878.

such *saturation*-values. By using more intense magnetic fields, on the contrary, it has been possible to force the induction to higher and higher values.

But if we consider the difference of induction and intensity, that is the expression $\mathfrak{B} - \mathfrak{H}$, it is apparent that this shows the characteristic course which the processes described present. It has even been established that one of the phenomena which is most easily and accurately accessible to quantitative determination, the rotation of the plane of polarisation, is in all circumstances proportional to $\mathfrak{B} - \mathfrak{H}$. It is probable, further, from all our present experimental observations, that the other phenomena also depend either on the difference $\mathfrak{B} - \mathfrak{H}$ or on its square $(\mathfrak{B} - \mathfrak{H})^2$, according as they are even or odd functions of $\mathfrak{B} - \mathfrak{H}$.

It is natural, therefore, to introduce this expression, or one proportional to it, as a measure of the magnetic condition, as manifested in the physical properties of the ferromagnetic substance. Following the historical development, and taking into consideration the usual absolute electro-magnetic system, we shall choose as factor the number $1/(4\pi)$ (compare note p. 59), and put therefore

$$(12) \quad . \quad . \quad . \quad . \quad \mathfrak{J} = \frac{\mathfrak{B} - \mathfrak{H}}{4\pi}$$

The value $\mathfrak{J}$ defined by this equation we briefly call the *magnetisation*:[1] it will in the sequel play an important part. By transforming the equation (12) we obtain the fundamental equation

$$(13) \quad . \quad . \quad . \quad . \quad \mathfrak{B} = \mathfrak{H} + 4\pi\mathfrak{J}$$

In the typical cases of ferromagnetic rings which have hitherto been considered, those three quantities $\mathfrak{B}$, $\mathfrak{H}$, and $\mathfrak{J}$ all have the same direction, so that the interpretation of equation (13) presents no difficulties. We shall subsequently (§ 51) return to its more general character as a vector equation.

In order to give some idea of the numerical values in question, the following data may be mentioned. In ordinary

[1] This abbreviated expression is already frequently used instead of intensity of magnetisation; the latter, besides being longer, has the disadvantage of being liable to be confounded with the intensity $\mathfrak{H}$.

circumstances the magnetic field of a coil cannot be driven beyond a few hundreds of electro-magnetic C.G.S. units. Only by using special arrangements for cooling (ice or currents of water) is it possible in this way to attain at most 1,500 C.G.S. The saturation-value of magnetisation in the case of iron, which as a ferromagnetic substance has by far the most extended application, is in the most favourable case from 1,700 to 1,750 C.G.S. units; the product $4\pi\mathfrak{J}$ is then between 21,500 and 22,000.

Hence, according to equation (13), it has but little influence if a few hundreds are added to the latter number. With small values of magnetisation the second term $4\pi\mathfrak{J}$ still more outweighs the first. We arrive, thus, at the practically important result, that in ordinary circumstances we may with close approximation put

$$(14) \quad \mathfrak{B} = 4\pi\mathfrak{J}$$

In those cases in which $\mathfrak{H}$ is intentionally raised to higher values than are attainable with simple coils, this simplification of course does not hold.

§ 12. **Summary.**—Once more summarising our results, the consideration of experimental facts leads to the following conception.

The physical condition of a ferromagnetic substance is completely defined by the magnitude which we have called magnetisation, and denoted by $\mathfrak{J}$.

The electromotive impulses in the surrounding conductors are not, however, defined by the magnetisation $\mathfrak{J}$, but by the induction $\mathfrak{B}$, for this reason, that they depend, not only on the condition of the ferromagnetic substance, but also on that condition which would hold if it were removed, or if it lost its specific properties. To illustrate this by an example, let us assume that the ring in question consisted of nickel, while the coils were of platinum wire insulated by asbestos. If this arrangement is heated above 300°, a change will be perceived; the specific ferromagnetic properties of nickel rapidly diminish and disappear altogether at 350°,[1] the metal becomes nearly

[1] This phenomenon also occurs with other ferromagnetic metals, but at a much higher temperature.

indifferent in the sense explained in § 8. The property of acting inductively on adjacent conductors also diminishes considerably, but does not disappear; in the expressions used above the quantity of electricity induced by opening or closing the primary current would on heating fall from Q' to Q, but not to zero.

In this state of things there would be no essential change on further rise of temperature, not even if the melting-point of nickel were reached, and the metal flowed away. By closing or opening the primary current, the quantity of electricity Q is always induced; it measures the condition which magnetisation produces in an indifferent medium or in heated nickel. But if we again cool the nickel, the ferromagnetic properties also return, and at the same time the value Q rapidly increases to the value Q'. We are accordingly compelled to admit that the specific condition, which alone produces those properties, is superposed on that condition which previously existed in the space in question, whether it was void or was occupied with indifferent matter. It is the superposition of the two conditions, the variations of which can under suitable conditions produce electromotive efforts.

From the great importance of this inductive action, the induction $\mathfrak{B}$ was long regarded, particularly by some English authors, as the more fundamental magnitude. Herein there has, however, already been a change of opinion, and now, as at first, the magnetisation is considered as the physically more important conception in dealing with purely scientific questions. The idea of induction of course loses none of its great value for the mathematical treatment of cognate problems on the one hand, and for technical applications on the other. We shall in the sequel have frequent opportunity to recognise this, when we enter on the field of applied physics (Chapter VI. *et seq.*).

§ 13. **Curves of Magnetisation. Curves of Induction.**—In a ring of any given dimensions, consisting of a definite ferromagnetic material, both $\mathfrak{J}$ and $\mathfrak{B}$ are only functions of $\mathfrak{H}$, apart from any question of hysteresis. The graphical representations of those functions for the case of rings are therefore to be considered as *normal curves*, since they only are characteristic for the material in question. For such a curve, as we shall see further on, is, in general, considerably, often, indeed, prepon-

deratingly influenced by the *form* of the ferromagnetic substance. As regards the experimental details of these curves, we must refer to works which treat magnetic induction with completeness.[1] We shall here only represent and discuss such a curve of ascending reversals for an iron ring; the curves for other ferromagnetic substances show always the same general character, even when they present marked deviations in their individual features.

Generally speaking three parts can be distinguished in all *normal curves of magnetisation*. The first corresponds to the smallest value of the magnetising intensity; the magnetisation then increases about in proportion to that vector, so that this portion deviates but little from a straight line through the origin of co-ordinates. The curve then bends strongly away from the abscissa axis, corresponding to a more rapid increase of magnetisation; this second portion corresponds to mean intensities. The curve then reaches a point of inflexion, increasing at last continually more slowly. The last portion, which corresponds to the highest intensities, even to infinite values, so far as at present known, gradually approaches the asymptote parallel to the axis of abscissæ; this represents a maximum value of magnetisation to which this quantity tends without strictly speaking ever attaining it. The three portions of the curve correspond to three different stages of the processes of magnetisation represented by it. And this division is not merely an arbitrary one, but is based on the very nature of the process. In special circumstances it is possible to put the ferromagnetic substance in a molecular condition in which the three stages of magnetisation are quite remarkably developed, and appear distinct from each other.[2]

Fig. 2 represents, for instance, the normal curve of magnetisation, $\mathfrak{J} = \text{funct.} (\mathfrak{H})$ for a variety of cast iron; the } indicates that for infinite values of the abscissa $\mathfrak{H}$ the ordinate $\mathfrak{J}$ would attain the saturation value $\mathfrak{J}_m = 1100$ C.G.S. units; it does, in fact, approach this asymptotically, if the intensity is more and more increased.

[1] Compare, for example, Ewing, *Magnetic Induction in Iron and other Metals*, chaps. iv. vi. and vii.

[2] Compare Nagaoka, *Journ. Coll. Science Imp. Univ. Japan*, vol. **2**, pp. 263, 304, 1888; *ibid.* vol. **3**, p. 189, 1889.

In order to transform a normal curve of magnetisation [$\mathfrak{J}$ = funct. ($\mathfrak{H}$)] into the corresponding *normal curve of induction* [$\mathfrak{B}$ = funct ($\mathfrak{H}$)], we take into account equation (13).

$$\mathfrak{B} = 4\pi\mathfrak{J} + \mathfrak{H}$$

in which the first term on the right usually very much exceeds the second; it will therefore in most cases be sufficient to read off the curve of magnetisation on a scale of $1/4\pi$ (right scale of ordinates fig. 2). In cases in which the approximation $\mathfrak{B} = 4\pi\mathfrak{J}$ is not sufficient, we have to add a portion to that

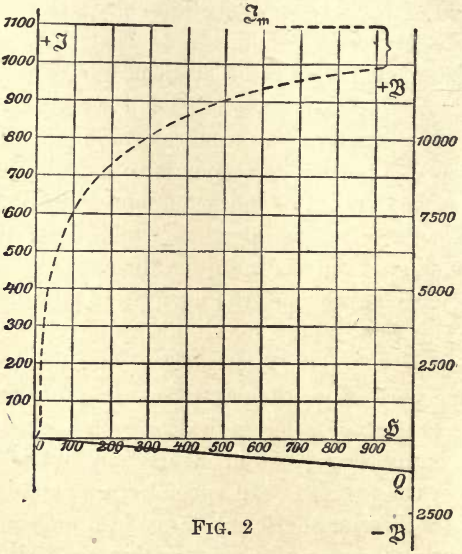

FIG. 2

ordinate, which is manifestly equal to the numerical value of the corresponding abscissa.

This is most conveniently done by a kind of transformation of co-ordinates; that is, by drawing through the origin a straight line $\overline{OQ}$, the equation of which is $-\mathfrak{B} = \mathfrak{H}$, and then measuring the ordinates from this new axis of abscissæ. If, however, we are to return to the ordinary orthogonal system of co-ordinates, we must distort the whole plane of the figure, which we assume to be completely extensible, in such a way that each point moves upwards parallel to the axis of ordinates, so that the straight line $\overline{OQ}$ ultimately coincides with the original axis of abscissæ.

Such an operation Ewing calls a *shearing* of the curve, parallel to the ordinates, from the directrix $\overline{OQ}$ to the axis of abscissæ. Such shearing of curves, which, however, is usually done parallel to the abscissæ, and whose directrices need not necessarily be straight, we shall frequently have to deal with for other purposes. The operation can be conveniently effected by means of a pair of compasses.

The normal curve of induction deduced from fig. 2 is represented in fig. 3. While, however, in the former, the abscissæ are plotted to $\mathfrak{H}=1000$, in the latter this is carried up to the value

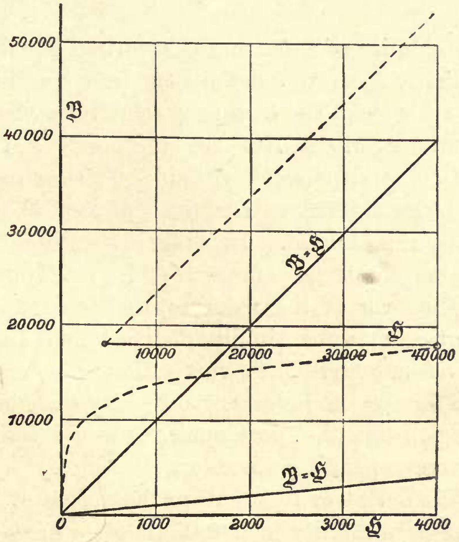

FIG. 3

of the intensity $\mathfrak{H}=40{,}000$, the highest which has hitherto been reached. For this reason it was necessary to use two scales of abscissæ. The first, which is already one-fourth that of fig. 2, extends to $\mathfrak{H}=4000$; the continuation of the curve can be easily read on the accessory axis by the $\frac{1}{10}$ scale. These curves will clearly show how the vector $\mathfrak{J}$ tends towards a maximum, whereas the vector $\mathfrak{B}$ increases finally without limit.

§ 14. **Susceptibility, Permeability, Reluctivity.**—Besides the chief quantities $\mathfrak{J}$ and $\mathfrak{B}$, both which, apart from hysteresis, are single-valued functions of $\mathfrak{H}$, some other less important

quantities have been introduced which are frequently useful. They are thus briefly defined:

κ, magnetic Susceptibility, defined as $\mathfrak{I}/\mathfrak{H}$
μ, ,, Permeability, ,, ,, $\mathfrak{B}/\mathfrak{H}$
ξ, ,, Reluctivity, ,, ,, $\mathfrak{H}/\mathfrak{B}$

These scalar quantities are all pure numbers—at least, they are thus to be regarded, in all cases in which the electro-magnetic system is used. In this system the vectors $\mathfrak{I}$, $\mathfrak{B}$, and $\mathfrak{H}$ have the same dimensions

$$[L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-1}]$$

and the permeability of a vacuum is supposed equal to unity, as follows implicitly from the definitions in question; the same holds for its reciprocal, the magnetic reluctivity of a vacuum.

These three numbers, like the vectors $\mathfrak{I}$ and $\mathfrak{B}$, are only functions of the independent variable $\mathfrak{H}$; they may, however, just as well be considered as functions of $\mathfrak{I}$ or $\mathfrak{B}$, and as such be graphically represented. In different authors we find, for instance, representations of $\kappa = \mathfrak{f}(\mathfrak{I})$, $\mu = \mathfrak{f}(\mathfrak{B})$, $\kappa = \mathfrak{f}(\mathfrak{H})$, $\xi = \mathfrak{f}(\mathfrak{H})$. The form of these curves, their singular points, and other properties may be simply deduced and discussed from those of the normal curve of magnetisation; they are interesting as guides in forming a judgment on many questions. But to enter here upon all the developments would lead us too far without any corresponding advantage.

We confine ourselves therefore to the curves of fig. 4, which represent the permeability μ (left scale of ordinates) and their reciprocal the coefficient of reluctivity ξ (right scale of ordinates) in so far as they depend on the intensity $\mathfrak{H}$. They refer to the same cast iron as figs. 2 and 3. The permeability, as will be seen, attains a maximum which may be very considerable. For the best wrought iron it may, for instance, attain a value of several thousands; it then falls off gradually until with an unlimited increase of $\mathfrak{H}$ it would presumably have the value 1. Conversely, the reluctivity passes through a sharply defined minimum, and then gradually increases, and would likewise at last approach the value 1, though far outside the region represented in fig. 4. (Compare § 121.)

If in equation (13) § 11

$$\mathfrak{B} = \mathfrak{H} + 4\pi\,\mathfrak{J}$$

both sides are divided by $\mathfrak{H}$ we obtain

$$\frac{\mathfrak{B}}{\mathfrak{H}} = 1 + 4\pi\frac{\mathfrak{J}}{\mathfrak{H}} \quad .$$

or if the above values are introduced

(15) . . . . $\mu = 1 + 4\pi\,\kappa$

as the relation between permeability and susceptibility. The latter number would probably vanish if the intensity increased indefinitely; but it is to be observed that of course only suppositions can be made for this case, based upon what actually happens, at the highest intensities attainable (§ 13). The susceptibility then appears as the quotient of a finite quantity, the saturation of the magnetisation, divided by an infinite quantity. We have therefore $\kappa_\infty = 0$; from equation (14) it follows that $\mu_\infty = 1$, and in like manner $1/\mu_\infty = \xi_\infty = 1$ as given above.

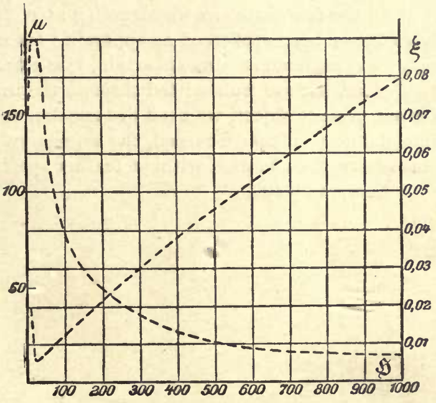

FIG. 4

As we shall afterwards make no distinction between magnetically indifferent bodies and a vacuum, we shall consequently suppose the permeability of the former unity, whereas it actually differs in the fourth or fifth decimal place. The reluctivity is then also unity, but by equation (14) the susceptibility is zero.

§ 15. **Perfect and Imperfect Magnetic Circuits.**—Our previous considerations referred to the case of a uniformly wound circular ring, having also a circular section; a body thus shaped we have called a toroid. But we may at once extend them to rings of any given section and any shape. The

centroid (§ 6), that is, the curve corresponding to the central circle of the ring (§ 9), may be either a plane or a solid curve, provided only that its radius of curvature be always large compared with the dimensions of the section. With uniform winding there will be no apparent magnetic action at a distance[1]—that is, the magnetic condition is restricted to the body of the ferromagnetic ring. Such an arrangement is called a *perfect magnetic circuit*. We assume that the condition for this perfection is the absence of any action at a distance, and hence an arrangement in which such an action does occur is aptly called an *imperfect magnetic circuit*.[2]

[1] In the few places in which action at a distance has been mentioned, it has always been spoken of as apparent; for in the present state of science it may be considered as almost certain, that direct actions at a distance do not take place, but are transmitted through the intervening medium. But since, for our present object, we need not concern ourselves with this deeper insight into the mechanism concerned, the expression 'action at a distance' will, with this reservation, be used without further specification.

[2] Compare § 100.

CHAPTER II

ELEMENTARY THEORY OF IMPERFECT MAGNETIC CIRCUITS

§ 16. **Action of a Narrow Transverse Cut.**—The reason for the comparatively simple behaviour of perfect magnetic circuits in the sense of the previous paragraph, is their geometrical property of having no ends, so that in the literal sense of the words, they are *endless*. This *endlessness* entails, as will be seen, the absence of action at a distance, and may therefore just as well be regarded as that property which directly determines the perfection of the magnetic circuit.

The correctness of this conception is seen when we provide the perfect circuit with ends by cutting it. Any cut, however fine, then manifests itself by the occurrence of an action at a distance, which is strongest in the immediate neighbourhood; in the part surrounding the cut magnetic conditions are produced, the intensity of which increases with the width of the cut. The occurrence of actions at a distance, conversely, points to the presence of transverse cuts, even when these cannot be seen, as for instance in the case of concealed cracks, or such as have been soldered, or joints in the ferromagnetic substance itself (Chapter IX., § 151).

The air-gap which occupies the cut differs in no respect from the indifferent environment, and therefore the action at a distance cannot proceed from it; we must accordingly conclude, as a geometrical necessity, that the action at a distance is due to the ends which the cross cut has produced.

In order to acquire an insight into the action of such cuts from a totally different point of view, let us consider the typical case of a uniformly wound toroid. We suppose this to be cut transversely in the direction of the radius say at A (fig. 1, p. 11), in such a manner that the cut is very narrow, compared with the dimensions of the section; let its width be d. Let (A) be the

normal curve of magnetisation of the closed ring before it has been cut, as represented in fig. 5 for a particular variety of soft iron. If then the curve is determined for the divided ring, it will be found to run more to the right, like the curve (B) for instance. We thus see that with a given magnetising intensity as abscissa, the ring which has been cut has a smaller magnetisation than the closed one, and that conversely, in order to produce the same magnetisation, a more powerful field is required. By how much must this be greater?

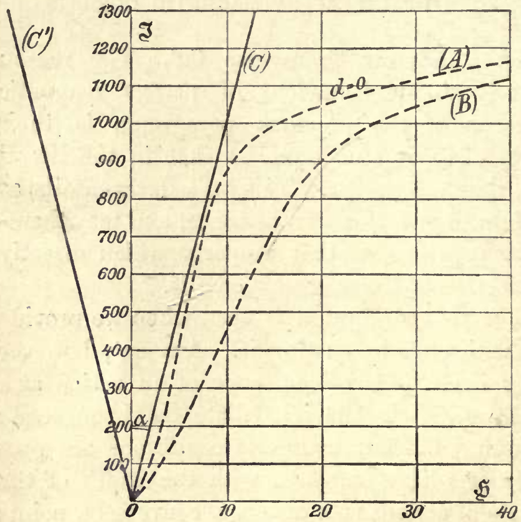

FIG. 5

To answer this question we have to determine for given values of the ordinates, the differences of abscissæ $\Delta\,\mathfrak{H}$ of the two curves, and then to try in what way these depend on the ordinates, that is to examine them as functions of the ordinates. For narrow transverse cuts we thus obtain, as a first approximation, a straight line $\overline{OC}$, which is so inclined that for each given ordinate

$$\text{Absc. }(C) = \text{Absc. }(B) - \text{Absc. }(A).$$

The answer to the question is, therefore, that to obtain a given magnetisation with divided rings, an increase of intensity is necessary, which, as a first approximation, is proportional to the magnetisation to be attained; that is to say, is a certain constant fraction of it.

The theory and experiments by which these results are arrived at will be discussed in detail in a special section (Chapter V.); we deal here only with the general qualitative results, in so far as they afford an insight into the action of cuts.

§ 17. **Shearing; Backward Shearing.**—This is the appropriate place for discussing the graphical operations by means of which, by transforming co-ordinates, normal curves of magnetisation may be transformed into curves for divided rings, or, in general, for imperfect magnetic circuits, and conversely. With this object we draw a straight line $\overline{OC'}$ through the origin O (fig. 5) into the upper left of the quadrant, which is symmetrical with $\overline{OC}$ with respect to the axis of ordinates. It is clear from what has been said above that we obtain the new curve (B) from the normal curve (A), when, in conformity with § 13, we effect a *shearing* from the directrix $\overline{CO'}$ to the axis of ordinates. By the converse operation, which we may call *backward shearing* from the axis of ordinates to the line $\overline{CO'}$, we transform again the curve (B) into the normal curve (A).[1]

The equation of the line $\overline{OC'}$ we may write generally

$$\Delta\mathfrak{H} = N\mathfrak{J}$$

in which N is a factor increasing with the width of the cut, to which we shall afterwards refer in more detail. The increase in the intensity of the magnetising field, $\Delta\mathfrak{H}$, which has to be applied in consequence of the cut, is, in fact, according to equation (1) proportional to the value $\mathfrak{J}$ of the magnetisation to be induced, as was also mentioned in the previous paragraph.

§ 18. **Action at a Distance of the Ends.**—As regards the explanation of the facts described, we have already seen that actions at a distance are directly produced by a cut, and the more markedly the wider is the cut. These actions extend over the whole surrounding space, and it is, therefore, natural to suppose that they will also be produced in the space occupied by the ferromagnetic substance of the ring, and will, therefore, act inductively. Now it may be readily shown that such an inductive action must be opposed to that of the field due to the current in the coil (fig. 6, p. 27), so that the latter must be increased, in order to compensate the former and thus to obtain the same total intensity, and, hence, the same magnetisation. It is this increase in intensity which is expressed by the difference of the abscissæ of the curves, and which, therefore, must be equal and

[1] This construction was first given by Lord Rayleigh, *Phil. Mag.*, 5th series, vol. **22**, p. 175, 1886, and generalised by Ewing.

opposite to the mean intensity of the action at a distance of the cut within the space occupied by the ring itself.

As, further, we have already seen that the air-gap itself can exert no action, and that this must proceed from the ends, we conclude that such a pair of ends acts at great distances proportionally to the value of the magnetisation. We have thus arrived at a conception which necessitates the closer investigation ot the action at a distance of pairs of ends.

We are, at the same time, led to a point of view which was formerly the general starting-point, and the one almost exclusively taken into consideration. A noteworthy reaction has, in some quarters, recently set in, which at once considers as inappropriate, antiquated, and useless the ideas of magnetic fluids, of poles, attractions, and Coulomb's law, &c., which were peculiar to that mode of view.[1] We shall proceed most safely in this respect if we take a middle course, and endeavour, as far as possible, to combine and utilise the advantages of both views.

§ 19. **Action at a Distance of a Single End.**—Let us imagine the ring cut through, and then stretched out so that it has the form of a long straight bar, and thus its ends are at the greatest distance possible. In the divided ring we assumed that the coiling was uniform, notwithstanding the gap. We shall here make the same assumption, and imagine a uniformly wound closed coil, like that in fig. 6, which does not produce of itself an external magnetic field, but within the windings produces a uniform field in the direction of the feathered arrows. Magnetisation in the same direction is then produced in the ferromagnetic bar, as can be proved by means of secondary coils closely wound round it. Owing to the action of the ends a magnetic field will now be formed in the surrounding space. We will now investigate this somewhat more closely, by means of an exploring coil as in § 2.

If now we investigate the space near one end, N for example, where its own influence decidedly preponderates, we shall find that the field has everywhere a radial direction, and from the end *outwards*, as shown by the unfeathered arrows. Its numerical value is inversely proportional to the square of the distance from

[1] Compare for instance, Prof. Silv. Thompson, *Cantor Lectures on the Electromagnet*, London, 1890.

the ends as long as this is small compared with the length of the bar. Near the other end the condition is the same, except that the radial field is directed *inwards* at the end, and is again represented by plain arrows. The latter (S) is usually called the negative, and the former (N) the positive, end.[1] The positive direction of magnetisation in ferromagnetic substances is always from the negative towards the positive end.

The intensity of the field near the ends increases proportionally to the value of magnetisation, as we proved with the divided ring, and is also proportional to the section S of the bar. It is independent of the length, provided this is considerable in comparison with the cross section, as is always supposed to be

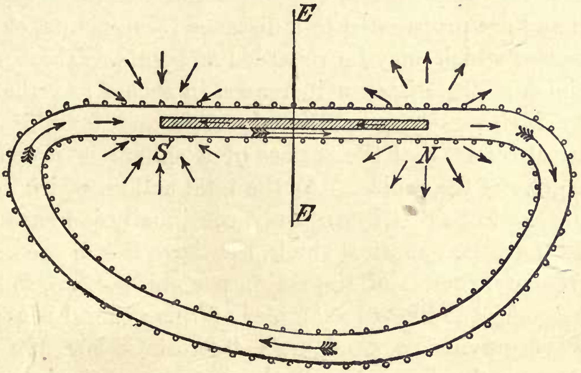

FIG. 6

the case. The absolute value of the intensity of the field is therefore given by the equation

(2) . . . . $\mathfrak{H} = \dfrac{\mathfrak{J} S}{r^2}$

where r is the distance from the end. Accordingly the power of the ends to exert actions at a distance depends upon the value of the product ($\mathfrak{J} S$), which may be called the *magnetic strength* of the bar.[2]

[1] With a freely suspended bar the positive end sets nearly north, the negative south. The positive end is by most writers called the north pole, the negative S, the south pole. (See Maxwell, *Treatise*, vol. **2**, § 393.)

[2] Sir W. Thomson, *Reprint of Papers on Electricity and Magnetism*, p. 354, § 454.

§ 20. **General Remarks about the Law of Action between Points.**—The relation expressed by equation (2) is a somewhat modified conception of what is called 'Coulomb's law,' to which we shall afterwards return. An essential condition for the validity of that law is, that the end of the bar is regarded as a point, and that, therefore, its dimensions are infinitely small compared with the distance r; the latter, again, is supposed to be small compared with the length of the bar. The experimental demonstration of the relations in question can therefore only be made with very long thin bars.

We may here observe that Coulomb's law is by no means a specific magnetic law. It is only a special case of the perfectly general, purely geometrical, law which governs all actions that proceed and are propagated to a distance from points, or rather from centres which may be regarded as points. These actions all diminish as the distance increases, in such a way that their intensity is inversely proportional to r^2, and for this simple geometrical reason that the surface of a sphere is proportional to the square of the radius. As the total action, which remains constant, has to be distributed over continually increasing surfaces of concentric spherical shells, the above law of the decrease of its intensity, that is of the action per unit surface, follows at once. Among the known examples of this general relation are the law of gravitation, Coulomb's electrostatic law, the photometric law for the decrease with the distance, of the intensity of light from luminous points. In all these cases the assumption of points as centres is the essential basis for the application of the $(1/r^2)$ *law of action between points or of inverse squares.*

A gravitating, an electrified, or a luminous infinitely long straight line, on the contrary, does not act inversely as the square of the distance, but inversely as the distance itself; and here again, for the simple reason that now the surface of a cylinder of given length is proportional to its radius. A further example of this general $(1/r)$ *law of the action between lines* is Biot and Savart's law of the electromagnetic action of long straight portions of conductors, which has been already discussed in § 5.

The action, finally, of a gravitating, electrified, or luminous infinite plane does not depend on the distance; it may, in

fact, be considered as enclosed by two planes, also infinite, the surface of which is obviously independent of the distance. For the sake of a general expression, we may also, in this case, speak of a $(1/r^0)$ *law of action between planes.*

§ 21. **Attraction or Repulsion between the Ends.**—The end of a magnet not only produces a field in its vicinity, but it is influenced by an existing extraneous field in such a manner that a mechanical force is exerted on it. This force has either the same or the opposite direction, according as that part of the original field where the end is placed is positive or negative. Its numerical value $\mathfrak{F}$ is equal in absolute measure to the product of the intensity of the field into the magnetic strength $(\mathfrak{J}\,S)$ of the end (§ 19):

$$(3) \quad . \quad . \quad . \quad . \quad \mathfrak{F} = \mathfrak{H}\,(\mathfrak{J}\,S)_1$$

If, as a particular case, the field in question radiates from another magnetic end, the resultant action may obviously be explained by saying that according as their signs are the same or opposite there is a repulsion or an attraction, $\mathfrak{F}_{12}$, between the ends, which is the direction of the line joining them, directly proportional to the product of their magnetic strengths, and inversely proportional to the square of the distance. The mathematical formula for this law follows directly from equations (2) and (3), and is as follows:

$$(4) \quad . \quad . \quad . \quad \mathfrak{F}_{12} = \frac{(\mathfrak{J}\,S)_1\,(\mathfrak{J}\,S)_2}{r^2}$$

This is the original form of Coulomb's law,[1] which was discovered by Coulomb by experiments with the torsion balance. An exact confirmation was first obtained by the measurements which Gauss made for this purpose.[2] As the existence of direct mechanical actions at a distance, which a verbal interpretation of this law entails, can scarcely be accepted by modern science, the tendency is to explain these apparent actions at a distance by stresses in the medium which transmits the action. We shall

[1] In the notation already mentioned (§ 19, note), it is usual to formulate Coulomb's law as follows. Like (unlike) poles, repel (attract) with a force which is directly proportional to the strength of the poles, and inversely proportional to the square of their distance.

[2] Gauss, *Intensitas vis Magn. Terrestr. ad Mensuram Absol. Revocata,* § 21, *Werke,* vol. 5, p. 109; 2nd reprint, Göttingen, 1877.

discuss these stresses in the gap more in detail in articles §§ 101–110. The introduction of this conception, besides affording a more satisfactory theoretical explanation of the facts, has the advantage of providing a far more suitable basis for most practical problems than Coulomb's law does. The latter, indeed, still gives the simplest representation of the mechanical actions in all cases in which it is a question of the reciprocal influence of a small number of ends of bars, as is still the case in many of the usual experimental methods.

§ 22. **Action at a Distance of a Pair of Ends.**—We will now return to the action at a distance of our bar magnet, and consider this at distances which are great compared with the length of the bar; the numerical value and direction of the magnetic field is obtained for all points by the superposition of the components due to the two ends as given from Coulomb's law. Two special cases are to be distinguished, the proof of which need not here be gone into.

In the first place, the intensity of the field at a point on the prolongation of the geometrical axis of the bar is directed along the axis, and its value is given by the equation

$$(5) \quad . \quad . \quad . \quad . \quad \mathfrak{H} = \frac{2\mathfrak{J}\,S\,L}{D^3}$$

in which L is the length of the bar, D the distance of the given point from the centre of the bar.

Secondly, at points in the equatorial plane of the bar, traced by the line EE, in fig. 6, the value is

$$(6) \quad . \quad . \quad . \quad . \quad \mathfrak{H} = \frac{\mathfrak{J}\,S\,L}{D^3}$$

where the direction is still parallel to the axis.

We thus see how the length of a bar now enters as a factor into the equations, and accordingly ($\mathfrak{J}\,S\,L$) and not ($\mathfrak{J}\,S$) is the measure of its power to produce a field at distant points. But $S\,L$ is the volume of the bar, and therefore the product of the magnetisation into the volume is the determining quantity. In accordance with the usual practice we shall call this the *magnetic moment* of the bar, and denote it by $\mathfrak{M}$; it is analogous to the corresponding quantity for coils, which is also thus designated (§ 6). If the action at a distance is used to measure

the magnetic properties of bodies—as in what are called magnetometric methods—the magnetisation is equal to the magnetic moment measured divided by the volume.

In case the point in question is neither in the prolongation of the axis nor in the equatorial plane of the bar, the equations for the intensity of the field are less simple than (5) and (6). As regards the shape of the lines of intensity we refer to fig. 7 in which they are represented outside the bar by the dotted lines; the figure in question holds for a short bar; with longer bars, as assumed in the above, the action at a distance proceeds almost exclusively from the geometrical ends, and the lines accordingly diverge radially. In fig. 7 these lines proceed not only from the end surfaces $+E$ and $-E$, but also partially at

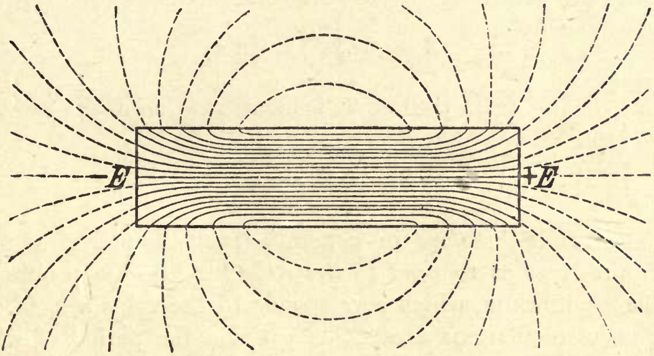

FIG. 7

least from the neighbouring parts of the sides. The lines either proceed from the positive to the negative end, or they direct their course towards an infinite distance.

§ 23. **Mechanical Action of External Fields on Pairs of Ends.**—We have already seen in § 21 how in an extraneous field a force is exerted on a single end. The occurrence of single ends is, however, excluded from the nature of the case; we can at most assume that in very long bars one of the ends may be considered singly by neglecting the actions proceeding from the other end, or exerted on it, as being too distant. But, as a matter of fact, in bar magnets, we have always to deal with a

[1] Compare § 26, where the lines inside the bar, 'the lines of magnetisation,' are further discussed.

pair of ends, whether the magnetisation is induced or residual. The area S and the magnetisation $\mathfrak{J}$ being constant, both ends have the strength $\mathfrak{J}\,S$, though with opposite signs. In an extraneous field, the intensity of which is constant within the space occupied by the bar, and is in the same direction, according to equation (3) § 21, equal and opposite forces are exerted on the ends.

$$\mathfrak{F} = \pm\,\mathfrak{H}\,(\mathfrak{J}\,S)$$

If the positive axial direction of the bar (§ 19) makes an angle a, with the positive direction of the field, these two forces act at points which are at a distance of $L \sin a$ from each other, L again being the length of the bar. They thus exert a torque, the moment of which is given by the equation

(7) . . . $$\mathfrak{K} = \mathfrak{H}\,(\mathfrak{J}\,S)\,L \sin a$$

or introducing again the magnetic moment $\mathfrak{M}$, as in the previous paragraph

(8) . . . $$\mathfrak{K} = \mathfrak{H}\,\mathfrak{M} \sin a$$

The couple therefore in general acts on the pair of ends in such a manner as to tend to draw the bar into the position of stable equilibrium, which corresponds to the value $a = 0°$. It will make oscillations about this position the period of which τ is given by the equation

(9) . . . $$\tau = 2\,\pi\sqrt{\frac{K}{\mathfrak{H}\,\mathfrak{M}}}$$

in which K is the moment of inertia of the bar. A position of unstable equilibrium corresponds to the value $a = 180°$.

These conclusions are most completely confirmed by experiment. A single force is never exerted on a magnet in a field of constant value, and constant direction, but always a torque; and this holds not only for bar magnets of constant strength, but also for bodies of any form, and magnetised in any way whatever.

§ 24. **Demagnetising Action of a Bar.**—It has already been explained in the case of the divided ring that each of the ends

will also act according to Coulomb's law in the space occupied by the bar itself. This 'self action,' as is at once seen from the plain and feathered arrows in fig. 6, p. 27, will always be opposed to the action of the coil. Both ends exert therefore a demagnetising action on the bar, the intensity of which we will call $\mathfrak{H}_i$. This is a minimum at the middle of the bar and increases towards each end; from what has been said, it is proportional at each point to the magnetic strength($\mathfrak{I}\, S$) of the bar. Hence this proportionality also holds for the mean value of the vector $\mathfrak{H}_i$, which we shall distinguish by a bar over the letter, thus $\overline{\mathfrak{H}}_i$.

Let us now consider a circular cylindrical bar with plane ends. The *dimensional ratio*, by which is meant the ratio of the length to the diameter, we will call $\mathfrak{m}$. If we now assume that the cylinder becomes gradually thinner, the length remaining the same, this would correspond to an increase of the ratio $\mathfrak{m}$, and to a decrease of the cross-section proportional to that of $1/\mathfrak{m}^2$. Hence $\overline{\mathfrak{H}}_i$ also decreases, and, from what has been said, in the same proportion as the magnetic strength—that is, proportionally to $(\mathfrak{I}/\mathfrak{m}^2)$.

If we now consider the quotient $\overline{\mathfrak{H}}_i/\mathfrak{I}$—that is, the mean demagnetising intensity per unit of magnetisation—and if, as in equation (1) (§ 17), we denote it with $\overline{N}$, we shall call the number thus defined the *mean demagnetising factor*. It follows, then, from the preceding that $\overline{N}$ must theoretically be proportional to $1/\mathfrak{m}^2$—that is,

I. *The demagnetising factor of a circular cylindrical bar is, theoretically, inversely proportional to the square of the ratio of the dimensional ratio.*

If C is the factor of the proportion, then

$$(9) \qquad \overline{N}\mathfrak{m}^2 = C$$

must be constant. From an analysis of experiments with cylinders, it is found that this, as a matter of fact, does hold, provided the ratio of the dimensions exceeds 100. C has then the constant value 45. The mean demagnetising factor of a cylinder, whose length is at least 100 times the diameter, may be simply calculated by dividing the square of the dimensional ratio into the number 45.

§ 25. **Demagnetising Factors of Circular Cylinders.**—The experiments mentioned were made with cylinders of varying lengths, but with a given diameter and given material. Owing to the unavoidable heterogeneity of the material, it is better to cut the cylinders gradually shorter from one and the same piece. It would be best of all, retaining the length constant, to gradually turn it down to a smaller diameter, as we have supposed in our theoretical investigation. The curves found thus experimentally for various values of $\mathfrak{m}$ are then plotted side by side, and the corresponding demagnetising factors [1] are then deduced from the differences of the abscissæ in the manner above described (§ 16).

It appeared that for short cylinders for which $\mathfrak{m} < 100$, the value C is not constant but diminishes. In Table I., which, for more convenient comparison with other numbers, is printed on page 41, a general view is given of the mean demagnetising factors found as described. The values $C = \mathfrak{m}^2 \overline{N}$ are also given, as being better fitted for interpolation owing to the slight extent to which they vary.

It has further been experimentally established that ferromagnetic prisms or bundles of wire tied together, of any given profile, differ but little from circular cylinders of equal length and equal section.[2] The table furnishes, therefore, a means of reducing the results of experiments with bars and bundles to the proper normal case of endless shapes; in other words, by back shearing, to obtain the normal curve of the material investigated, which is alone characteristic. This is the more necessary as by far the greater number of the existing and, in part, very valuable experiments have been made with bars; and from the nature of the case this will, in the future, also often occur. The results and the curves obtained in this way will also be difficult of direct interpretation, and hence lose much of their value.

§ 26. **Short Cylinder. Lines of Magnetisation.**—We have already seen that for short cylinders (for which $\mathfrak{m} < 100$) the

[1] Such experiments have been made partly by Ewing, *Phil. Trans.* vol. **176**, p. 535, partly by Tanakadaté, *Phil. Mag.* vol. **26**, p. 450, 1888. The theoretical deductions are due to the author (*Wied. Ann.* vol. **46**, p. 497, 1892).

[2] Von Waltenhofen, *Wien. Ber.* vol. **48**, part 2, p. 578, 1863. [Extended series of experiments on bundles of wires have lately been undertaken by Ascoli and Lari. *R. Acad. dei Lincei*, Rome, 1894. H. d. B.]

principle laid down in § 24 no longer holds, but that their demagnetisation factor is smaller than that principle prescribes; one or more of the assumptions on which it was deduced do not, therefore, hold.

In fact, if a secondary coil is displaced along the cylinder, and the momentary current excited in it on magnetisation is investigated, we find that this does not suddenly cease when the coil is pushed beyond the ends, but becomes smaller at a certain distance from the ends. This distance is relatively more important in the case of a short cylinder than in that of a long one. We conclude from this that the induction is greater in the middle of the bar than towards the ends. This, therefore, is also the case with the magnetisation.

If we examine the action at a distance due to the ends, we shall find it smaller than would correspond to the value of magnetisation in the middle of the bar. At the same time the distribution of the external field is of the same nature as if an action proceeded not only from the geometrical ends, but also from the adjacent parts of the bar: as if in a certain sense there were ends there also. We are, then, thus led to generalise the idea of 'ends.' We no longer consider it as a purely geometrical conception in the sense of meaning by it the terminal faces of the bar; we rather conceive the magnetisation of the short cylinder as if the stronger magnetic condition which exists in the middle of the bar only gradually ended, and therefore shows a great number of magnetic *end-elements*, each of which exerts its own elementary action at a distance in accordance with Coulomb's law. Just as a positive and a negative end were previously distinguished, so we must now distinguish as positive and negative the end-elements, which are distributed on the corresponding halves of the bar. In this the algebraic sum of the strengths of all the end-elements is always zero, even with a body of any given shape, and magnetised in any given way. This follows, among other things, from the fact adduced in § 23, that an external field of constant value and invariable direction exerts a torque on such a body and never a single force.

Closely connected with this is the course of the *lines of magnetisation*—that is, of those lines whose tangent at each point coincides in direction with the vector $\mathfrak{J}$, as was established

before for the lines of intensity for the vector $\mathfrak{H}$. A group of such lines of magnetisation has already been sketched in fig. 7, p. 31, with their approximate course inside a short cylinder. It will be seen that many lines do not terminate in the base, but do so previously in the cylindrical surface. These terminal points represent in a tangible manner, what at first sight is an unfamiliar idea, namely the end-element which we arrived at above.

We have hitherto only regarded systems of lines in general, and especially of lines of magnetisation, as a means of representing the directions of the vector considered, but we shall afterwards see that in many cases they also allow conclusions to be drawn concerning the numerical value of the vector.

§ 27. **The End-elements as Centres of Action at a Distance. Hypothesis of Two Fluids.**—We have thus arrived at the conception of a magnetic body as possessed of end-elements each of which appears to act at a distance according to Coulomb's law. Such apparent centres of action at a distance preponderate on the surface of bodies, and in some cases are confined to them; but, as we shall see, they occur in the most general case in the interior also. With the introduction of an infinite number of end-elements the problem in question passes, of course, from the region of elementary treatment to that of the higher mathematics. In this respect we must refer to Chapters III. and IV., although, in what follows, we discuss a few special cases, the analysis of which can only be given afterwards; we are here concerned only with results.

The conception at which we have arrived by two distinct methods is connected with a classical hypothesis, still widely diffused, to which we must devote a few words. Two magnetic fluids were assumed, a positive (north) and a negative (south), each elementary particle of which exerts an action at a distance according to Coulomb's law. These are distributed in exactly equal quantities over the magnetised body, to the greatest extent over the surface, but also in the interior. The mathematical treatment of these principles has led to results which are capable of exact experimental confirmation. In consequence of this, there has long been a disposition to regard the assumptions, which were the starting points, as also correct, without being restrained by the fact that those conceptions were from the outset devoid

of any plausible physical basis, having been developed at a time in which comparatively small weight was laid on such a thing.

The theory of north and south fluids here sketched has great similarity with the conception according to which end-elements must be regarded as the essential agent for actions at a distance. The older mathematical theory of Poisson may therefore be used just as before, without essential modification, as results from the following chapters; the adjustment of that theory to actual facts is chiefly the work of Lord Kelvin, F. Neumann, Kirchhoff, and Maxwell.

We have no occasion in this book to go minutely into the various hypotheses which from time to time have been made to explain the essential nature of magnetism; we may, however, mention that, before Poisson's theory, Euler put forth the hypothesis that magnetism is matter flowing in closed paths (§ 111). With our present views, neither hypothesis is probable. The theory which best suits the facts, and is most capable of development, is Weber's assumption of elementary particles whose directions are controllable, as further developed chiefly by Maxwell, Wiedemann and Ewing;[1] in this the pre-existing magnetism of the elementary particles may be either due to vortices or to Ampèrian molecular currents, or as assumed by Richarz and Chattock, to rotating ions.

§ 28. **Uniform Field. Ellipsoid.**—A magnetic field throughout a definite space is said to be uniform when it has everywhere the same value and the same direction; the lines of force are then parallel and straight. It is usual, in like manner, to speak of the uniform distribution of any vectors in space (§ 43). In our most recent considerations we have assumed uniform fields, as, for instance, that which a coil like the one in fig. 6, p. 27, produces in the interior. Such coils are scarcely used in practice, but straight solenoids at least three times as long as the body to be magnetised; in the space occupied by this latter in the middle of the coil the field is then sufficiently uniform for most purposes.

The magnetisation induced in a uniform field is not necessarily also uniform, as we have already seen in the case of the short cylinder. It may be mathematically proved (§ 69) that such

[1] Wiedemann, *Elektricität*, 3rd edition, vol. 3, 1883; Ewing, *Magnetic Induction*, chap. xi. Berlin, 1892.

uniformity only holds for bodies of an ellipsoidal shape[1]; if one axis coincides with the direction of the field, the magnetisation is in the same direction as that axis. The demagnetising intensity for unit magnetisation we shall again speak of as the demagnetising factor for the axial direction in question.

The demagnetising intensity in the interior of the body (§ 24) is in this case also uniformly distributed, as is also the resultant intensity due to this and to the original field of the coil (§ 68). The ellipsoid, and the geometrical shapes directly derivable from it, then become simple types of imperfect magnetic circuits, as the closed toroid is for perfect circuits.

§ 29. **Ellipsoid of Revolution: Ovoid; Spheroid.**—We shall here confine ourselves to the case of an ellipsoid of revolution, which can be turned in the lathe, and is therefore practically important.

Let $2c$ be the length of the axis of revolution, $\overline{CC'}$ (figs. 8, 9), $2a$ that of the equatorial diameter at right angles to this axis. The ratio of the axes is then $\mathfrak{m} = c/a$.

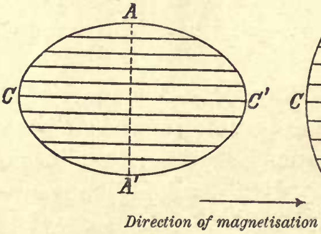

FIG. 8

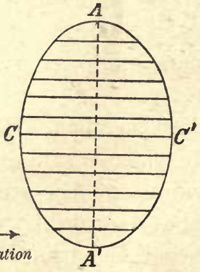

FIG. 9

It is assumed that the axis of revolution is parallel to the direction of the field, so that, from what has been stated above, the magnetisation is also in this direction. We distinguish two cases according as the axis of rotation is longer or shorter than the equatorial axis $\overline{AA'}$.

A. *Ovoid.* Prolate ellipsoid of revolution (elongated

[1] Compare also Maxwell, *Treatise*, 2nd edition, vol. 2, §§ 437, 438. The formulæ to be afterwards given are there deduced.

ellipsoid of revolution) $c > a$, consequently $\mathfrak{m} > 1$. The eccentricity of the meridian ellipse is given by the expression

$$\mathfrak{e} = \sqrt{1 - \frac{a^2}{c^2}} = \sqrt{1 - \frac{1}{\mathfrak{m}^2}}$$

N_2, the demagnetising factor in the direction of the axis of revolution, is given as a function of the eccentricity by the equation

$$(10) \quad . \quad N_z = 4\pi\left(\frac{1}{\mathfrak{e}^2} - 1\right)\left(\frac{1}{2\,\mathfrak{e}}\log\frac{1+\mathfrak{e}}{1-\mathfrak{e}} - 1\right)$$

or, if we take the ratio of the axes $\mathfrak{m}$ as the argument,

$$(11) \quad N_z = \frac{4\pi}{\mathfrak{m}^2 - 1}\left(\frac{\mathfrak{m}}{\sqrt{\mathfrak{m}^2 - 1}}\log\left(\mathfrak{m} + \sqrt{\mathfrak{m}^2 - 1}\right) - 1\right)$$

If $\mathfrak{m}$ in this expression has greater values, and we have to deal with more elongated ellipsoids, it approaches (as is at once seen if 1 is negligible as compared with $\mathfrak{m}^2$) the simpler form

$$(12) \quad . \quad . \quad N = \frac{4\pi}{\mathfrak{m}^2}(\log 2\,\mathfrak{m} - 1)$$

which, with ratios of the axes exceeding 50, gives values correct to within a few thousandths, and with still more elongated ovoids, almost absolutely correct values for N_z.

B. *Spheroid* (oblate ellipsoid of revolution) $c < a$, consequently $\mathfrak{m} < 1$. The eccentricity $\mathfrak{e}$ of the meridian ellipse is given by the expression

$$\mathfrak{e} = \sqrt{1 - \frac{c^2}{a^2}} = \sqrt{1 - \mathfrak{m}^2}$$

and the demagnetising factor as a function of this eccentricity by

$$(13) \quad . \quad N_z - \frac{4\pi}{\mathfrak{e}^2}\left(1 - \sqrt{\left(\frac{1}{\mathfrak{e}^2} - 1\right)}\ \text{arc sin}\ \mathfrak{e}\right)$$

or again as a function of the ratio $\mathfrak{m}$ by

$$(14) \quad . \quad N_z = \frac{4\pi}{1 - \mathfrak{m}^2}\left(1 - \frac{\mathfrak{m}}{\sqrt{1 - \mathfrak{m}^2}}\ \text{arc cos}\ \mathfrak{m}\right)$$

§ 30. **Further Special Cases. Solid Sphere and Cylinder. Plate.**—Mathematical analysis has hitherto succeeded in solving accurately only a few special cases of imperfect magnetic circuits (compare § 70). This want is, to some extent, com-

pensated by the fact that several other important shapes may be regarded as special cases of the ellipsoid.

C. *Solid Sphere.*—This may be regarded as an ellipsoid with three equal axes. We have, then, for the demagnetising factor in any direction,

$$(15) \quad . \quad . \quad . \quad . \quad N = \frac{4\pi}{3}$$

D. *Circular Cylinder* (transversely magnetised).—If we consider a ferromagnetic cylinder, which is magnetised not in the axial direction (as in § 24), but transversely, as an ellipsoid with two equal and a third infinitely long transverse axis, we have then for the demagnetising factor in any transverse direction, the value

$$(16) \quad . \quad . \quad . \quad . \quad N_x = 2\pi$$

E. *Thin Plate* (magnetised transversely).—The greatest demagnetising factor is possessed by a thin ferromagnetic plate, magnetised at right angles to its plane. It can obviously be regarded as an infinitely flattened spheroid, the eccentricity of which is then $\mathfrak{e} = 1$. By inserting this value in equation (13) we get

$$(17) \quad . \quad . \quad . \quad . \quad N_z = 4\pi.$$

§ 31. **Tabular Summary.**—In the special forms last discussed we have limited ourselves to a mere statement of the demagnetising factors, which in all these cases are constant over the whole body, and completely determine the problem of its magnetisation. When this factor is known, we can at once construct the curve of magnetisation for the special shape, as we shall presently see. Although the ellipsoid and its varieties may be taken as simple types of imperfect magnetic circuits, it is not usual to lay great stress on their properties as such, especially if they are considered in connection with the magnetic condition in the surrounding indifferent medium; for the considerations are, in this case, by no means simplified.

We must dispense with a more minute discussion of the magnetic relations which may be deduced from the values of N given by the above equations, and of which some are interesting in connection with experimental methods. Table I. gives a summary view of the demagnetising factors of cylinders (§ 25)

and ellipsoids of revolution for values of the ratio of dimensions (that is, of the ratio of the axes) between 0 and ∞. In each case, also, the products $C = \mathfrak{m}^2 \overline{N}$ or $= \mathfrak{m}^2 N$, as the case may be, are added, which lend themselves better to interpolation. It is apparent from the series of numbers that N is always greater for ovoids than the $\overline{N}$ of the corresponding cylinder. It is therefore inadmissible to make them equal, as has formerly been done.[1] An ovoid acts rather like a cylinder which is somewhat shorter.

TABLE I.—DEMAGNETISING FACTORS OF CYLINDERS AND ELLIPSOIDS OF REVOLUTION

m	Cylinder		Ellipsoid of Revolution		
	$C = \mathfrak{m}^2 \overline{N}$	$\overline{N}$	N	$C = \mathfrak{m}^2 N$	Special Case
0	0	12·5664	12·5664	0	Thin plate
0·5	—	—	6·5864	—	Spheroid
1	—	—	4·1888	—	Solid sphere
5	—	—	0·7015	—	Ovoid
10	21·6	0·2160	0·2549	25·5	,,
15	27·1	0·1206	0·1350	30·5	,,
20	31·0	0·0775	0·0848	34·0	,,
25	33·4	0·0533	0·0579	36·2	,,
30	35·4	0·0393	0·0432	38·8	,,
40	38·7	0·0238	0·0266	42·5	,,
50	40·5	0·0162	0·0181	45·3	,,
60	42·4	0·0118	0·0132	47·5	,,
70	43·7	0·0089	0·0101	49·5	,,
80	44·4	0 0069	0·0080	51·2	,,
90	44·8	0·0055	0·0065	52·5	,,
100	45·0	0·0045	0·0054	54·0	,,
150	45·0	0·0020	0·0026	58·3	,,
200	45·0	0·0011	0·0016	64·0	,,
300	45·0	0·00050	0 00075	67·5	,,
400	45·0	0·00028	0·00045	72·0	,,
500	45 0	0·00018	0·00030	75·0	,,
1,000	45·0	0·00005	0·00008	80·0	,,
∞	—	0	0	—	Endless

§ 32. **Graphical Representation.**—It follows from Table I. that the demagnetising factor increases from 0 to 4π (= 12·5664) if, starting from an endless figure (whether an infinitely long

[1] Compare W. Weber, *Elektrodyn. Maassbestimmungen*, vol. **3**, p. 573, 1867; Kirchkoff, *Ges. Abh.* p. 221; Oberbeck, *Pogg. Ann.* vol. **135**, p. 84, 1868. [Recent unpublished experiments, however, render it probable that the values of $\overline{N}$ in the table are correct for wire bundles, but rather too small for iron bars. H. d. B.]

one or a closed annular one), we gradually shorten it until we have ultimately a thin plate, in which the influence of the ends—that is, of the bounding surfaces on the two sides—manifestly attains its greatest possible value.

We may apply this to the curves of magnetisation, representing the influence of the demagnetising factor by the directrix $\Delta \mathfrak{H} = N \mathfrak{I}$ [equation (1), § 17], and performing the shearing process explained above. It is then manifest how, as the values of N increase, the directrix will be continually more inclined to the left from the axis of ordinates. The curve of magnetisation will therefore appear more displaced towards the right, the greater N is—that is, the shorter the ellipsoids, or the greater the gap in the rings. It is advisable to perform this construction for a concrete case, for a wrought-iron ellipsoid, for instance, and not on too small a scale. It will then be seen that, as the ellipsoid is shortened, the curve gradually alters its shape, and soon seems closely embraced by two straight lines. The first passes through the origin, and makes with the axis of ordinates the same angle towards the right as the directrix does towards the left. The second straight line proceeds parallel to the axis of abscissæ at the distance $\mathfrak{I}_m$, where $\mathfrak{I}_m$ is the maximum magnetisation. We see now how the first straight line is determined by the value of N alone—that is, by the shape—and the second by the nature of the material alone, but the point of intersection by both these factors. This holds also almost exactly for those parts of the curve which lie close to each straight line; by this property the curve will be at once held for an hyperbola.

§ 33. **Hyperbolic Curves of Magnetisation.**—For shorter ellipsoids, or for rings with wider gaps, in short for greater values of the factor N, the curve of magnetisation indeed differs but little from a hyperbola, the asymptotes of which are the straight line in question, and the equation of which may be most simply written

$$(18) \qquad x = Ny + \frac{P}{1-y}, \text{ or } x = \frac{Ny + (P - Ny^2)}{1-y}.$$

Here P denotes a second constant; the maximum magnetisa-

[1] Compare fig. 21, p. 131, which represents the influence of cuts of increasing width on the course of the curves of magnetisation.

tion is the unit of length; the equation may be simply deduced by the methods of analytical geometry.

It must be kept in mind that this is only an approximation to the true form of the curve. For the inclined part can never be the branch of an hyperbola, because it has a point of inflexion, though not very marked, and it must pass through the origin. As regards the horizontal branch, according to certain optical experiments, the law of its approach to the asymptote is, in fact, hyperbolic, at any rate in very intense fields.[1] This curve is not to be confounded with O. Frölich's purely empirical curve of 'effective magnetism,' to which we shall return in the eighth chapter.

We see thus how, as the demagnetising factor increases, the curves of magnetisation for ellipsoids or divided rings gradually recede from the axis of ordinates; at the same time the second asymptote parallel to the axis of abscissæ always remains the same. The limiting curves are, on the one hand, the normal curve ($N = 0$), and that for a thin plate, magnetised transversely, on the other ($N = 4\pi$). The former characterises the material and is of cardinal importance; the latter demands a somewhat separate position, since by its means various physical properties (§ 10) have become accessible to quantitative reasoning.

The whole of the curves obtained with bodies of different shapes of a given ferromagnetic substance must lie between these two limiting curves, and this holds not only for the two shapes, ellipsoids of revolution, and divided rings, but can be extended to any shape whatever in which the material is formed. The magnetisation is thus indeed no longer uniform, but what has been said applies to its mean component as before. The mean demagnetising factor for each shape may be empirically constructed from the difference of abscissæ of the corresponding curve found empirically, and the normal curve (compare Chapter V.)

[1] This appears to follow from optical measurements by the author (*Phil. Mag.* vol. **29**, p. 302, 1890), even if it cannot in general be considered established with certainty.

CHAPTER III

OUTLINES OF THE THEORY OF RIGID MAGNETS

A. *Geometrical theory of Vector-distribution*

§ 34. **Vector-distributions.**—In the beginning of the first chapter it seemed appropriate to premise some elementary considerations respecting quaternions; here also we must deal, as far as necessary, with geometrical considerations of a general character. Their introduction will produce greater clearness, in treating what is to follow, than would be attainable by the use of purely analytical methods.[1] We will first agree upon the conventions as to the notation used.[2]

The quaternion expressions scalar and vector we have already defined (§ 3); we shall denote a vector magnitude in general without reference to its special nature by $\mathfrak{F}$. In order to prevent the excessive accumulation of symbols to be used, suffixes will be added, such as x, y, z, ν, τ, in order to indicate the components of a vector in the directions of the X, Y, Z axes, and its components, *normal* and *tangential*, to given surfaces. The former is geometrically speaking the projection of the vector on the normal to the surface, the latter that on the tangent plane to the same at the point in question. The normal to a surface we shall generally denote by $\mathfrak{N}$; it has to be determined in each case which of its two directions shall be positive. The angle between two vectors is expressed by enclosing them in brackets and separating them by a comma. S will stand for areas of surfaces, and

[1] In the following Chapters III. IV. and V. we must abandon the use of elementary methods.

[2] We may here refer to two chapters of a general character in the first volume of Maxwell's *Treatise*; the introduction and chapter iv.

L for lengths of curves, $d\,S$ and $d\,L$ being the corresponding elements. The meaning of this notation is sufficiently shown by the following expressions :

$$\mathfrak{F}_\nu = \mathfrak{F} \cos (\mathfrak{F}, \mathfrak{N})$$

and

$$\mathfrak{F}_\tau = \mathfrak{F} \sin (\mathfrak{F}, \mathfrak{N})$$

which at the same time indicate that the numerical value of a vector-component is obtained by multiplying that of the vector itself into the cosine of the angle between the two directions (§ 3).

If we now consider a limited, or an infinitely extended region, within which a vector-quantity has a finite value, this may be called the field of the vector. Its direction, and its numerical value, will in general vary continuously from point to point; the existence of surfaces of discontinuity is not, however, excluded.

The region may be either simply- or multiply-connected; the former is always assumed unless the contrary is expressly mentioned. The manner in which the direction and value of the vector at a point are related to the position of the point is called the geometrical distribution of the vector in the region to be investigated. There are several different modes of distribution characteristic of the vector quantities occurring in nature. Each of these is conditioned by definite relations the analytical expression of which involves the derived functions (differential coefficients) of the vector-components with respect to the co-ordinates. It will be our immediate object to investigate these more thoroughly; but we have first to prove an important general principle.

§ 35. **Surface-Integrals and their Properties.**—The double integral

$$\mathbf{S} = \iint \mathfrak{F} \cos (\mathfrak{F}, \mathfrak{N})\, d\,S = \iint \mathfrak{F}_\nu\, d\,S$$

taken over S, is called the *surface-integral* of the vector $\mathfrak{F}$ over this portion S. It is obtained by multiplying each surface element into the component of the vector perpendicular to it, and integrating this product over the entire surface. Let us specially consider the surface-integral over a *closed* surface S'.

Within the space enclosed, let $\mathfrak{F}_x$, $\mathfrak{F}_y$, $\mathfrak{F}_z$, be continuous and finite, except at a surface of discontinuity, $F(x, y, z)$ where these vector components experience an abrupt change of value; let their values on one side of the surface be simply denoted by $\mathfrak{F}_x$, $\mathfrak{F}_y$ and $\mathfrak{F}_z$, and those on the other side by $\mathfrak{F}'_x$, $\mathfrak{F}'_y$, and $\mathfrak{F}'_z$.

Let the positive direction of the normal $\mathfrak{N}_s$ to the closed surface be always that which is drawn inwards; we shall for shortness denote its direction cosines by

$$\mathfrak{l}_s = \cos(\mathfrak{N}_s, X), \quad \mathfrak{m}_s = \cos(\mathfrak{N}_s, Y), \quad \mathfrak{n}_s = \cos(\mathfrak{N}_s, Z)$$

In like manner we write the direction cosines of the normal to F

$$\mathfrak{l}_f = \cos(\mathfrak{N}_f, X), \quad \mathfrak{m}_f = \cos(\mathfrak{N}_f, Y), \quad \mathfrak{n}_f = \cos(\mathfrak{N}_f, Z)$$

If now we draw a straight line parallel to the X axis, this in general must cut the surface S' in an even number of points; first assume that there are only two, and that between these the straight line cuts the surface of discontinuity F. Following this line in the positive direction, it will first enter the surface S' at any point $x = x_1$, at which $\mathfrak{F}_x = \mathfrak{F}_{x1}$, and

$$\mathfrak{l}_s \, dS' = dy\,dz$$

it will then cut the surface F, where $\mathfrak{F}_x$ changes abruptly to the value $\mathfrak{F}'_x$; further,

$$\mathfrak{l}_f \, dF = dy\,dz$$

the straight line finally will emerge from S' at some point $x = x_2$, where $\mathfrak{F}_x = \mathfrak{F}_{x2}$ and

$$\mathfrak{l}_s \, dS' = -\,dy\,dz$$

We have then, as is known,

$$(1) \quad . \quad . \qquad \mathfrak{F}_{x2} - \mathfrak{F}_{x1} = \int_{x_1}^{x_2} \frac{\partial \mathfrak{F}_x}{\partial x}\,dx + (\mathfrak{F}'_x - \mathfrak{F}_x)$$

We are now in a position to calculate our surface integral; for from the well-known equation

$$\cos(\mathfrak{F}, \mathfrak{N}) = \cos(\mathfrak{F}, X)\cos(\mathfrak{N}, X) + \cos(\mathfrak{F}, Y)\cos(\mathfrak{N}, Y) + \cos(\mathfrak{F}, Z)\cos(\mathfrak{N}, Z)$$

it follows that

$$(2) \quad . \quad \iint \mathfrak{F} \cos(\mathfrak{F}, \mathfrak{N}_s)\, d\,S' = \iint \mathfrak{F}_x\, \mathfrak{l}_s\, d\,S' + \iint \mathfrak{F}_y\, \mathfrak{m}_s\, d\,S' + \iint \mathfrak{F}_z\, \mathfrak{n}_s\, d\,S'$$

in which all the double integrals are to be taken over S'. Let us now consider one of the members on the right, the first, for example. We observe that, in accordance with the above,

$$\iint \mathfrak{F}_x\, \mathfrak{l}_s\, d\,S' = \iint (\mathfrak{F}_{x1} - \mathfrak{F}_{x2})\, dy\, dz$$
$$= -\iiint \frac{\partial\, \mathfrak{F}_x}{\partial\, x}\, dx\ dy\ dz + \iint (\mathfrak{F}_x - \mathfrak{F}_x')\, \mathfrak{l}_f\, d\,F$$

If we introduce this, together with the similar expressions for the two other members, we obtain finally from (2)

$$(3) \quad \begin{aligned} \iint \mathfrak{F} \cos(\mathfrak{F}, \mathfrak{N}_s)\, d\,S' &= -\iiint \left(\frac{\partial\, \mathfrak{F}_x}{\partial\, x} + \frac{\partial\, \mathfrak{F}_y}{\partial\, y} + \frac{\partial\, \mathfrak{F}_z}{\partial\, z}\right) dx\ dy\ dz \\ &+ \iint \{(\mathfrak{F}_x - \mathfrak{F}_x')\, \mathfrak{l}_f + (\mathfrak{F}_y - \mathfrak{F}_y')\, \mathfrak{m}_f + (\mathfrak{F}_z - \mathfrak{F}_z')\, \mathfrak{n}_f\}\, d\,F \end{aligned}$$

the expression

$$-\left(\frac{\partial\, \mathfrak{F}_x}{\partial\, x} + \frac{\partial\, \mathfrak{F}_y}{\partial\, y} + \frac{\partial\, \mathfrak{F}_z}{\partial\, z}\right)$$

we may call, with Maxwell, the convergence of the vector at the point considered.

If now we state equation (3) in words we arrive at the following fundamental theorem.

I. *The surface-integral of a vector over a given closed surface is, apart from discontinuities, equal to the volume-integral of the convergence of the vector over the whole region enclosed.*

If there is a surface of discontinuity, a term must be added which on closer consideration may be described as follows: it is the surface-integral of the difference of the normal components of the vector on the two sides of the surface of discontinuity integrated over that portion of the latter which is enclosed by S'.

Our straight line parallel to the axis of X may moreover cut the surface S' in more than two points, and any number of surfaces of discontinuity may occur without the proof being essentially different. We have reproduced the general proof simplified as

much as possible, for the development here given can scarcely be considered to be as well known as the principle itself.[1]

§ 36. **Complex Solenoidal Distribution.**—A method suitable for exhibiting the distribution of a vector in space consists in supposing *vector lines* to be introduced—that is, curves at each point of which the tangent line is coincident in direction with the vector. We have already used this method more than once, having introduced lines of intensity (§ 4) and lines of magnetisation (§ 26). Change of direction along such curves will generally be continuous, but at surfaces of discontinuity they may be abruptly bent, or may terminate.

We may further consider the field as divided into wider or narrower tubes fitting closely against each other, the surfaces of which have the curves in question for their generating lines. These figures, which we shall call *vector-tubes*, will then extend over the whole of the region in question, their direction and their sectional area being in general variable from point to point. These variations may in general be quite unrestricted; such a vector-distribution of the most general kind may be called a *complex solenoidal* one[2] (from σωλήν, tube).

In dealing with the physical vector quantities which occur in nature we meet with modes of distribution in which the variations of the cross-section, and of the direction of the vector-tubes, are restricted by special relations, to the consideration of which we will now turn our attention.

§ 37. **Solenoidal Distribution.**—There is in the first place an important kind of distribution in which the surface-integral of the vector over any given closed surface is zero. A glance at equation (3) shows that this is equivalent to two other necessary

[1] For further mathematical details it will be sufficient to refer to Maxwell (*Treatise*, vol. 1, § 21, Theorem III), which states the theorem, together with the proof in the form given. and names, as the discoverer in 1828, the Russian mathematician Ostrogradsky (*Mém. de l'Acad. de St.-Pétersbourg*, vol. 1, p. 39, 1831). This important theorem is connected, moreover, with the equation of continuity, as we shall presently see, and may be regarded as a special case of Green's more general theorem which was made known in the same year (Green, *Essay on the Application of Mathematics to Electricity and Magnetism*, Nottingham, 1828).

[2] Sir W. Thomson, *Reprint of Papers on Electricity and Magnetism*, § 509; from which source most of what immediately follows is taken.

and sufficient conditions. In the first place, throughout the space enclosed by the surface we must have

$$(4) \quad . \quad . \quad . \qquad \frac{\partial \mathfrak{F}_x}{\partial x} + \frac{\partial \mathfrak{F}_y}{\partial y} + \frac{\partial \mathfrak{F}_z}{\partial z} = 0$$

that is, the convergence of the vector at each point must vanish. In the second place, at all surfaces of discontinuity

$$\mathfrak{l}_f \mathfrak{F}_x + \mathfrak{m}_f \mathfrak{F}_y + \mathfrak{n}_f \mathfrak{F}_z = \mathfrak{l}_f \mathfrak{F}'_x + \mathfrak{m}_f \mathfrak{F}'_y + \mathfrak{n}_f \mathfrak{F}'_z$$

or, what is the same thing,

$$(5) \quad . \quad . \quad . \quad . \qquad \mathfrak{F}_\nu = \mathfrak{F}'_\nu$$

The former equation (4) may be called the *volume-equation of continuity*, since in hydrodynamics, and in the theories of diffusion and of thermal and electrical conduction, it expresses the condition for the continuity of the corresponding components of flow (Chapter VII.) In like manner equation (5) is to be regarded as the *boundary-equation* of continuity.

We shall now suppose our integration to extend over a surface S' of special form, choosing for this purpose the surface of a finite vector-tube, whose ends are closed by surfaces S_1 and S_2 (fig. 10). The intermediate portion of the surface enclosed by the vector lines we shall call S_0; its share in the value of the surface-integral **S** is obviously zero, since that surface is everywhere tangential to the vector, and the latter has consequently no component perpendicular to the surface. The whole surface-integral is then seen to be the sum of portions furnished by the end-surfaces, which we designate by $\iint_{S_1}$ and $\iint_{S_2}$. But now, from the assumption at the beginning of this paragraph,

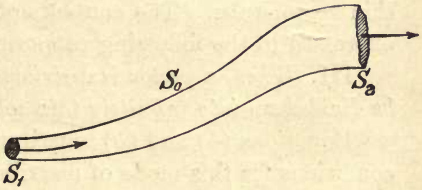

FIG. 10

$$\mathbf{S} = \iint_{S_1} + \iint_{S_0} + \iint_{S_2} = 0$$

but since $\iint_{S_0} = 0$, as explained above,

$$\iint_{S_1} = -\iint_{S_2}$$

As we previously assumed that the direction of the normal drawn inwards was positive, the sign of the term on the right side is to be reversed if now, on the contrary, we consider the direction of the vector as positive at both surfaces S_1 and S_2 (fig. 10).

In the mode of distribution here considered, the whole region may be divided up into vector-tubes, which have the following property:

II. *The surface-integral of the vector is the same over every cross-section of the tube.*

Such a vector-tube is called a simple *solenoid*, and the corresponding distribution a *solenoidal* one.

Let us now consider an infinitely thin vector-tube, so that the value of the vector does not vary appreciably over the cross-section, and draw the surfaces S_1 and S_2 at right angles to the direction of the tube. If then the values of the vector at the two terminal surfaces are $\mathfrak{F}_1$ and $\mathfrak{F}_2$, we may write the two portions of the surface-integrals as follows:—

$$\iint_{S_1} = \mathfrak{F}_1 S_1 \text{ and } \iint_{S_2} = \mathfrak{F}_2 S_2$$

The product $(\mathfrak{F}\ S)$ may be called the strength of the infinitely thin vector-tube. The contents of this paragraph may then be expressed in the following proposition:—

III. *When a vector is distributed solenoidally, the field may be divided up into infinitely thin solenoids of constant strength.*

Equations (4) and (5) constitute the necessary and sufficient conditions for this mode of distribution.

In a thin vector-tube of constant strength $(\mathfrak{F}\ S)$, the numerical value of the vector is obviously inversely proportional to the normal cross-section. A solenoid, the strength of which is everywhere unity, may be called a *unit-tube*; its section is everywhere numerically equal to the reciprocal of the vector; hence the greater the value of the latter, the more unit-tubes will be cut by a given surface. The 'density' with which the unit-tubes are crowded together in space furnishes a direct measure of the value of the vector, since the number which falls on a normal surface of unit area is numerically equal to the mean value of the vector over that surface.

§ 38. **Complex Lamellar Distribution.**—As regards the variation of the direction of the vector from point to point, we premise that in general it is not possible to construct a system

of surfaces such that at each point the vector is perpendicular to the surface through that point; that is to say, that the pencil of vector lines is throughout orthogonal to the system of surfaces. The necessary and sufficient condition for the existence of such an orthogonal system of surfaces is expressed by the well-known equation

$$(6)\ \mathfrak{F}_x\left(\frac{\partial \mathfrak{F}_y}{\partial z}-\frac{\partial \mathfrak{F}_z}{\partial y}\right)+\mathfrak{F}_y\left(\frac{\partial \mathfrak{F}_z}{\partial x}-\frac{\partial \mathfrak{F}_x}{\partial z}\right)+\mathfrak{F}_z\left(\frac{\partial \mathfrak{F}_x}{\partial y}-\frac{\partial \mathfrak{F}_y}{\partial x}\right)=0$$

which is also the condition that the equation

$$\mathfrak{F}_x\, dx + \mathfrak{F}_y\, dy + \mathfrak{F}_z\, dz = 0$$

may be integrable.[1]

The terms on the left may be transformed by a scalar integrating divisor $f\,(x, y, z)$ into an exact differential, the integral of which $\Theta\,(x, y, z)$, equated to a number of constant parameters Θ_1, Θ_2, Θ_3, represents the system of surfaces in question—that, namely, which is orthogonal to the pencil of vector lines.

The surfaces corresponding to successive values of the parameter enclose shell-like spaces, into which the region in question may be divided; such figures are said to be *complex shells.* They have only the geometrical property that at every point the vector meets them at right angles; its value has generally no connection with their thickness. Distributions of the kind here considered are said to be *complex lamellar.*

§ 39. **Lamellar Distribution.**—If the above-mentioned integrating divisor $f\,(x, y, z)$ is equal to unity—in other words, if the expression

$$\mathfrak{F}_x\, dx + \mathfrak{F}_y\, dy + \mathfrak{F}_z\, dz$$

is already an exact differential—the distribution belongs to an important special group. The necessary and sufficient conditions for the immediate integrability of the above expression are given by the equations

$$(7)\quad . \quad . \quad \frac{\partial \mathfrak{F}_y}{\partial z}=\frac{\partial \mathfrak{F}_z}{\partial y},\ \frac{\partial \mathfrak{F}_z}{\partial x}=\frac{\partial \mathfrak{F}_x}{\partial z},\ \frac{\partial \mathfrak{F}_x}{\partial y}=\frac{\partial \mathfrak{F}_y}{\partial x}$$

which in hydrodynamics are the equations characteristic of irrotational motion. The integral of the exact differential taken with reversed sign is called the scalar *potential* of the vector; we shall denote it by Φ.

[1] Schlömilch's *Handbuch der Mathematik,* vol. 2, p. 871, *et seqq.* Breslau, 1881. [Forsyth, *Differential Equations,* §§ 151, 152.]

We have accordingly

$$(8) \quad . \quad . \quad d(-\Phi) = \mathfrak{F}_x\, dx + \mathfrak{F}_y\, dy + \mathfrak{F}_z\, dz$$

or

$$(9) \qquad \mathfrak{F}_x = -\frac{\partial \Phi}{\partial x}, \quad \mathfrak{F}_y = -\frac{\partial \Phi}{\partial y}, \quad \mathfrak{F}_z = -\frac{\partial \Phi}{\partial z}$$

Since the potential is constant over any one of the orthogonal surfaces in conformity with the analytical results given above, in this case they are called *equipotential surfaces.* If their normal in the direction of increasing potentials is called $+\mathfrak{N}$, then from what has been stated, and because in this case the vector itself is perpendicular to the equipotential surface, and therefore is identical with its normal component,

$$\mathfrak{F} = \mathfrak{F}_\nu = -\frac{\partial \Phi}{\partial \mathfrak{N}}$$

If we consider an infinitely thin shell-like space between the two infinitely near equipotential surfaces Φ and $\Phi + d\Phi$, and if the portion of the normal intercepted between them—that is, the thickness of the shell—is called $d\mathfrak{N}$, then throughout the entire shell

$$\mathfrak{F}\, d\mathfrak{N} = d\Phi = \text{constant}$$

The thickness at any point is inversely proportional to the value of the vector, since the product of the two quantities, which is called the strength of the shell, is everywhere constant. Such a shell is called a simple shell, and the corresponding vector-distribution a *lamellar* one. We may express these considerations by the following theorem :—

IV. *When a vector is distributed lamellarly, its field is divided into infinitely thin shells of constant strength.*

This lamellar character is bound up with the equations (7), which at the same time condition the existence of a scalar potential.

In a thin vector-shell of constant strength ($\mathfrak{F}\, d$), the numerical value of the vector is inversely proportional to the variable thickness d. A shell whose strength is constant and equal to unity may be called a *unit-shell*; its thickness is everywhere numerically equal to the reciprocal of the vector; hence the greater the value of the latter, the more unit-shells will be in-

tercepted on a given length at right angles to the equipotential surfaces. The 'density' with which unit-shells succeed each other is a direct measure of the value of the vector, as the number intercepted on a line of unit length, drawn perpendicularly to the equipotential surfaces, is numerically equal to the mean value of the vector over that length.

§ 40. **Line-Integrals and their Properties.**—The definite integral

$$ {}^{B}_{A}\mathbf{L} = \int_A^B \mathfrak{F} \cos(\mathfrak{F}, L)\, d\,L = \int_A^B \mathfrak{F}_L\, d\,L $$

taken on the curve L between two points A and B is called the line-integral of the vector $\mathfrak{F}$, along the length $\overline{AB}$. It is obtained by multiplying each element of the curve into the tangential component of the vector, and integrating this product over the whole length.

If we consider A as the starting-point, and define B (whose co-ordinates we may call x, y, z) by its distance L from the point A, as measured along the path of integration, then, considering the equation

$$ \cos(\mathfrak{F}, L) = \cos(\mathfrak{F}, X)\cos(L, X) + \cos(\mathfrak{F}, Y)\cos(L, Y) + \cos(\mathfrak{F}, Z)\cos(L, Z) $$

we may write

(10) . . $$ {}^{B}_{A}\mathbf{L} = \int_0^L \mathfrak{F}\cos(\mathfrak{F}, L)\, d\,L $$

$$ = \int_0^L \left(\mathfrak{F}_x \frac{\partial x}{\partial L} + \mathfrak{F}_y \frac{\partial y}{\partial L} + \mathfrak{F}_z \frac{\partial z}{\partial L}\right) d\,L $$

The value of this expression differs, in general, with the path of integration by which we pass from A to B. If, however, the distribution of the vector is everywhere lamellar, so that

$$ \mathfrak{F}_x\, dx + \mathfrak{F}_y\, dy + \mathfrak{F}_z\, dz = d\,(-\Phi) $$

it follows from equation (10) that under all circumstances

(11) . . $$ {}^{B}_{A}\mathbf{L} = \int_0^L \frac{\partial \Phi}{\partial L} d\,L = \Phi_A - \Phi_B $$

That is, expressed in words:—

V. *When the distribution of a vector is lamellar, its line-integral is equal to the difference of the potentials at the two*

terminal points, independently of the path of integration between those points.

It follows directly from this that in the case of a lamellar distribution the line-integral taken round any closed curve must vanish. For imagine any two given points on the curve, the portions of the integral corresponding to the two paths between them are equal and opposite, and accordingly their sum is zero.

In these investigations it is tacitly assumed that the region considered is a simply-connected one. If the distribution in a multiply-connected region is lamellar, while at external points the vector does not possess this property, the potential becomes, in general, a multiple-valued function of the co-ordinates, and the theorems mentioned only hold under certain restrictions. This is not the place to pursue any further general investigations, whose interest would be chiefly theoretical.[1] We will, however, mention one special case, which we shall often meet with in the sequel. It is that of an annular space in which a vector has a lamellar distribution. The line-integral of the vector is not zero around a closed curve, which makes a complete circuit of the annular space, but with the completion of each circuit the integral increases by a constant, which is independent of the position of the curve within the ring so long as no part lies in the external space in which the distribution is supposed to be no longer lamellar.

§ 41. **Lamellar-solenoidal Distribution.**—A vector can be distributed both lamellarly and solenoidally, provided it satisfies the two necessary conditions. The first property implies the existence of a potential Φ; that is, in accordance with the foregoing,

$$\mathfrak{F}_x = \frac{\partial \Phi}{\partial x}, \quad \mathfrak{F}_y = \frac{\partial \Phi}{\partial y}, \quad \mathfrak{F}_z = \frac{\partial \Phi}{\partial z}$$

If we insert these values in the equation of continuity [§ 37, equation (4)], which is the condition that the vector may possess the second property, we obtain

$$(12) \quad . \quad . \quad . \quad \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

[1] See Helmholtz, *Crelle's Journal*, vol. 55, p. 25, 1858; *Wiss. Abhandl.* vol. 1, p. 101; Maxwell, *Treatise*, introduction; Lejeune-Dirichlet, *Vorlesungen über*

Accordingly, when the vector $\mathfrak{F}$ is distributed in this lamellar-solenoidal manner, the potential must satisfy the last-written equation, which expresses the necessary and sufficient condition for such a distribution. This equation, which is of great importance in all branches of physics, is often written for shortness $\nabla^2 \Phi = 0$, and is called *Laplace's equation.* Maxwell also applies the name of 'Laplacian distribution' to such a distribution of the vector $\mathfrak{F}$ as satisfies equation (12).

At surfaces where discontinuity arises, the *normal* component of the vector must preserve its continuity as we pass from one side of the surface to the other, just as in the case of a simple solenoidal distribution.

Since the distribution is solenoidal, the cross-section of the vector-tube is at each point inversely proportional to the value of the vector; and since the distribution is also lamellar, the same holds true for the distance between consecutive equipotential surfaces. The system of equipotential surfaces, together with the orthogonal system of vector-tubes, divide the field into 'cells,' whose volume is inversely proportional to the square of the vector.

§ 42. **Complex Lamellar-solenoidal Distribution.**—We have already seen that, in the case of a complex lamellar distribution, the expression

$$\mathfrak{F}_x\, dx + \mathfrak{F}_y\, dy + \mathfrak{F}_z\, dz$$

can be rendered integrable by a scalar integrating divisor; that is, it can thus be made an exact differential $d\,\Theta$. We must therefore have

$$\frac{\mathfrak{F}_x}{f} = \frac{\partial \Theta}{\partial x}, \quad \frac{\mathfrak{F}_y}{f} = \frac{\partial \Theta}{\partial y}, \quad \frac{\mathfrak{F}_z}{f} = \frac{\partial \Theta}{\partial z}$$

If we insert these values in the equation which conditions the solenoidal property [§ 37, equation (4)], we obtain

$$(13) \quad . \quad f\nabla^2 \Theta + \frac{\partial \Theta}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial \Theta}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial \Theta}{\partial z}\frac{\partial f}{\partial z} = 0$$

as the necessary and sufficient condition to be satisfied by the parameter Θ, in order that the vector $\mathfrak{F}$ may have a complex lamellar-solenoidal distribution.

die im umgekehrten Verhältniss des Quadrats der Entfernung wirkenden Kräfte, Leipzig, 1876; Clausius, *Die Potentialfunktion und das Potential.*

§ 43. **Uniform Distributions. General Laws.**—Of the distributions referred to above, several are extremely simple in character, and many examples of these will be found in the sequel. A vector is said to be distributed uniformly when at every point throughout the region considered it has the same direction and the same value. We may further distinguish the following cases:—

In a hollow shell bounded by two concentric spherical surfaces there is said to be an accurately (or appreciably) *uniform radial* distribution when the vector is everywhere in the direction of the radius, and has a strictly (or appreciably) constant value. It is easy to see that such a distribution is lamellar, the equipotential surfaces being those of concentric spheres. On the other hand, the distribution is *not* solenoidal, but falls infinitely little short of possessing this quality when we come to consider an infinitely thin spherical shell.

In the case of a toroid [anchor-ring (§ 9)], a vector is said to have an accurately (or appreciably) *uniform peripheral* distribution when the direction of the vector is at each point along the circumference of that circle in which the point would move if the toroid were rotated about its axis of symmetry, the value of the vector being at the same time accurately (or appreciably) constant. This kind of distribution is complex lamellar-solenoidal, since the vector is directed along lines orthogonal to 'meridian planes' through the axis of the anchor-ring. When the ring is infinitely thin, the distribution falls infinitely little short of being lamellar.

The analytical conditions for the above distributions are so simple that it seems unnecessary to write them *in extenso* or to give a full investigation of them.

In conclusion we may adduce some general laws concerning the distribution of a vector:—

VI. *If a vector satisfies any of the above conditions, its product with a scalar constant satisfies the same conditions.*

VII. *The superposition of two or more solenoidal or lamellar distributions yields a resultant distribution, which is in turn solenoidal or lamellar, as the case may be.*

For it is evident from the above that the derivatives of the components of the vector with respect to the co-ordinates are only

involved linearly in the equations conditioning these two kinds of distribution. In the case of a complex lamellar distribution this is, however, no longer the case, and consequently the law of superposition does not, in general, hold good for such distributions.

B. *Conductors conveying Currents, and Rigid Magnets*

§ 44. We must now consider the principal properties of the electromagnetic field which is produced by currents flowing in conductors. This has already been introduced as the foundation for all further treatment, and we have also (§§ 5, 6) given the electromagnetic formulæ for the special cases of most importance. We must here content ourselves with merely writing down some systems of equations which are not of an elementary character, referring for a fuller treatment of them to those works which deal exhaustively with electromagnetism.[1]

Consider a conductor of any form, and let $\mathfrak{C}$ denote the current-flow at a given point (x, y, z) of the conductor—that is, the current per unit area of normal cross-section at the point in question.[2] It will be convenient in this place to introduce the auxiliary vector $\mathfrak{A}$ (Maxwell's 'vector potential'), having components of the following values at any point distant r from the first-named point, namely—

$$(14) \quad . \quad . \quad \begin{cases} \mathfrak{A}_x = \iiint \frac{\mathfrak{C}_x}{r}\, dx\, dy\, dz \\ \mathfrak{A}_y = \iiint \frac{\mathfrak{C}_y}{r}\, dx\, dy\, dz \\ \mathfrak{A}_z = \iiint \frac{\mathfrak{C}_z}{r}\, dx\, dy\, dz \end{cases}$$

the integration being extended throughout the whole volume of the conductor.

It can then be shown that the components of the electromagnetic field due to the currents in the conductor can be expressed in terms of the derivatives of the auxiliary function $\mathfrak{U}$ with respect to the co-ordinates; they are, in fact, given by

[1] For example, Maxwell, *Treatise*, vol. 2, part iv.; Mascart and Joubert, *Electricity and Magnetism*, vol. 1, part iv.

[2] Often called 'current density,' and expressible in ampères per square centimeter.

the following equations, which hold good for all points, whether within or without the conductor :—

$$(15) \quad . \quad . \quad \begin{cases} \mathfrak{H}_x = \dfrac{\partial \mathfrak{A}_z}{\partial y} - \dfrac{\partial \mathfrak{A}_y}{\partial z} \\ \mathfrak{H}_y = \dfrac{\partial \mathfrak{A}_x}{\partial z} - \dfrac{\partial \mathfrak{A}_z}{\partial x} \\ \mathfrak{H}_z = \dfrac{\partial \mathfrak{A}_y}{\partial x} - \dfrac{\partial \mathfrak{A}_x}{\partial y} \end{cases}$$

The magnetic intensity $\mathfrak{H}$ arising from the electric current is now distributed solenoidally in all parts of the field without exception; for, by differentiating the above equations, we have identically

$$\frac{\partial \mathfrak{H}_x}{\partial x} + \frac{\partial \mathfrak{H}_y}{\partial y} + \frac{\partial \mathfrak{H}_z}{\partial z} = 0$$

and this is the condition for a solenoidal distribution of the vector $\mathfrak{H}$ [§ 37, equation (4)]. It can also be shown that $\mathfrak{H}$ satisfies the superficial equation of continuity at the surface separating the conductor from the surrounding medium.

§ 45. **Magnetic Potential in the Field outside Conductors.**— In close connection with the equations of the last paragraph we have the following :—

$$(16) \quad . \quad . \quad \begin{cases} 4\pi\,\mathfrak{C}_x = \dfrac{\partial \mathfrak{H}_z}{\partial y} - \dfrac{\partial \mathfrak{H}_y}{\partial z} \\ 4\pi\,\mathfrak{C}_y = \dfrac{\partial \mathfrak{H}_x}{\partial z} - \dfrac{\partial \mathfrak{H}_z}{\partial x} \\ 4\pi\,\mathfrak{C}_z = \dfrac{\partial \mathfrak{H}_y}{\partial x} - \dfrac{\partial \mathfrak{H}_x}{\partial y} \end{cases}$$

which also hold good for all points throughout the field.

If the point considered lies within the conductor, where $\mathfrak{C}$ has a finite value, these equations show that the distribution of $\mathfrak{H}$ cannot there be lamellar, since a potential does not exist. But outside the conductor, where there is no current, and where, consequently, $\mathfrak{C}$ is zero, the terms on the left will vanish, and the terms on the right then express the condition that the vector $\mathfrak{H}$ may be distributed lamellarly (the condition, that is,

that its components may be the space-derivatives of a potential). We have, in fact,

$$\frac{\partial \mathfrak{H}_y}{\partial z} = \frac{\partial \mathfrak{H}_z}{\partial y}, \quad \frac{\partial \mathfrak{H}_z}{\partial x} = \frac{\partial \mathfrak{H}_x}{\partial z}, \quad \frac{\partial \mathfrak{H}_x}{\partial y} = \frac{\partial \mathfrak{H}_y}{\partial x}$$

[Compare § 39, equation (7).] We accordingly arrive at the following theorem :—

VIII. *In the region surrounding a conductor* (not within the substance of the conductor itself) *the magnetic intensity is distributed lamellarly, and has consequently a scalar potential.*

This potential of the magnetic intensity $\mathfrak{H}$ we shall denote by Υ, and we shall call it the *magnetic potential.* But the region where $\mathfrak{H}$ is lamellar is now no longer simply connected, since it is interrupted by a non-lamellar region at least doubly connected, which includes every separate closed conducting circuit round which a current is flowing. Thus the present case differs from that previously discussed (§ 40). In fact, the line-integral of $\mathfrak{H}$ taken along any closed curve increases by a 'cyclic constant of integration,' C, for each time that the closed curve embraces the conductor, and it is only along such closed curves as do not embrace the conductor that the line-integral vanishes. In using the term 'cyclic constants' we must, of course, understand that these can only depend on the electric currents embraced by the path along which the line-integral is taken, and not on any purely geometrical relations. We can now determine *à priori* the value of C by observing that its relation to the current I must be one of proportionality, since the same is known from experiment to be true of $\mathfrak{H}$. The completion of the value of C can involve nothing further than the introduction of a numerical factor, and to this again, as in a former case (§ 11), we assign the value 4π,[1] following the historical development of the subject, and in conformity with the system of electromagnetic units in general use. Hence, finally,

(17) . . . $C = 4\pi I.$

[1] The introduction of the factor 4π is condemned by several authors, especially by Heaviside. Its elimination, however, could only be effected by remodelling the system of units already adopted, so that this comparatively unimportant simplification would be rather dearly bought; moreover, the factor would probably reappear in another place.

We can easily verify this equation from one of the elementary examples already considered; we shall choose the case of a long straight linear conductor, in the neighbourhood of which the electromagnetic action follows the law of Biot and Savart. The magnetic intensity $\mathfrak{H}$ at a distance r from the conductor is [§ 5, equation (3)]

$$\mathfrak{H} = \frac{2\,I}{r}$$

The line of force through the point considered is a circle of circumference $2\,\pi\,r$; and consequently the value of the line-integral of $\mathfrak{H}$ along a circuit embracing the conductor once (that is, the value of the cyclic constant C which we have to determine) is

$$C = \int_0^{2\,\pi\,r} \mathfrak{H}\,d\,L = \frac{2\,I \cdot 2\,\pi\,r}{r} = 4\,\pi\,I$$

a conclusion identical with (17). We can combine these results in the following fundamental theorem:

IX. *Every time the circuit along which we integrate embraces a conductor conveying a current I, the line-integral of magnetic intensity increases by the cyclic constant $4\,\pi\,I$, independently of any other variables whatever.*

This important general law is abundantly confirmed by experiment, as in the present case is almost self-evident.

§ 46. **Action of a Rigid Magnet at External Points.**—We have already (§ 26) introduced the conception of magnetic end-elements, referring to the present chapter for a fuller mathematical development of the method. We have also given [in § 19, equation (2)] a statement of the elementary law of Coulomb, that the force apparently exerted by a magnetic pole, at any point in its neighbourhood, is given by

$$\mathfrak{H} = \frac{\mathfrak{J}\,S}{r^2}$$

where $\mathfrak{H}$ is the magnetic intensity at the distance r from the end, and is directed away from the end when the end is positive, towards the end in the contrary case. S is the cross-sectional area of the bar-magnet, and $\mathfrak{J}$ the vector which we have called the magnetisation (§ 11).

Let us consider now, instead of a simple bar-magnet, a ferro-magnetic body of any shape, magnetised in any arbitrary manner without reference to the cause of such magnetisation. When we wish to express this total independence of the magnetisation on external causes, we shall speak of such a body as a *rigid magnet*; the phenomenon of magnetic retentiveness shows us the possibility of realising such a rigidly magnetised body with some degree of approximation. Within the region occupied by the body, therefore, we are to consider the distribution of the vector $\mathfrak{I}$ to be entirely arbitrary (and in general, consequently, complex-solenoidal, § 36); at the bounding surface the component along the normal drawn inwards is $\mathfrak{I}_\nu$, while along the normal drawn outwards into the magnetically indifferent surrounding medium, the component of magnetisation is zero. To calculate the magnetic intensity at external points, we may divide the body up into elementary parallelopipeds $dx\, dy\, dz$, write down the expression for the effect due to the three pairs of opposite faces of an infinitely small parallelopiped, and integrate this expression throughout the whole volume of the body.

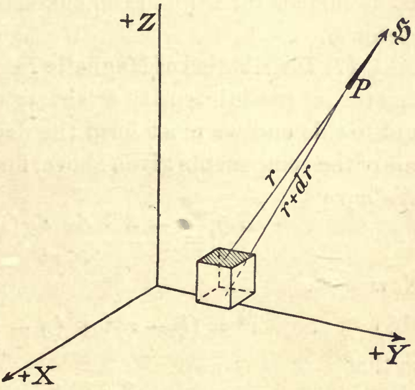

Fig. 11

Let us consider the parallelopiped $dx\, dy\, dz$ as a short bar parallel to the axis of z; then the component of magnetisation in a direction parallel to this axis becomes $+\mathfrak{I}_z$. Since the magnetisation is a vector, we make what use we please of the principle of resolution into components. Let us now fix our attention on the upper face of the parallelopiped (shaded in fig. 11), its co-ordinates being x, y, z, and the area of the face in question $dx\, dy$; its magnetic strength, which determines its influence at external points, is therefore $+\mathfrak{I}_2\, dx\, dy$, by the definition already given (§ 19).

At a point P (ξ, η, ζ), whose distance from (x, y, z) is r, this

end-element will exert the magnetic intensity $\mathfrak{H}$ given by the equation

$$\mathfrak{H} = + \frac{\mathfrak{J}_z\, dx\, dy}{r^2} .$$

and the components of $\mathfrak{H}$ will have the values

$$\mathfrak{H}_x = \frac{\mathfrak{J}_z\, dx\ dy\, (\xi - x)}{r^3}, \qquad \mathfrak{H}_y = \frac{\mathfrak{J}_z\, dx\ dy\, (\eta - y)}{r^3},$$

$$\mathfrak{H}_z = \frac{\mathfrak{J}_z\, dx\ dy\, (\zeta - z)}{r^3}$$

Before combining the field due to the above end-element face with that due to the opposite face of the infinitesimal parallelopiped, let us further investigate the character of the distribution of the vector $\mathfrak{H}$.

§ 47. **Distribution of Magnetic Intensity.**—We shall first prove that the magnetic intensity $\mathfrak{H}$ is everywhere lamellarly distributed, and to this end we must form the derivative with respect to ζ of one of the components given above, for example the Y-component. We have

$$\frac{\partial\, \mathfrak{H}_y}{\partial\, \zeta} = \frac{-\, 3\, \mathfrak{J}_z\, dx\ dy\, (\eta - y)}{r^4} \frac{\partial r}{\partial \zeta}$$

Now

$$(18) \quad . \qquad r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2$$

so that

$$\frac{\partial\, r}{\partial\, \zeta} = \frac{\zeta - z}{r}$$

Inserting this value in the above equation, we obtain

$$\frac{\partial\, \mathfrak{H}_y}{\partial\, \zeta} = \frac{-\, 3\, \mathfrak{J}_z\, dx\ dy\, (\eta - y)\, (\zeta - z)}{r^5}$$

In the same manner

$$\frac{\partial\, \mathfrak{H}_z}{\partial\, \eta} = \frac{-\, 3\, \mathfrak{J}_z\, dx\ dy\, (\zeta - z)\, (\eta - y)}{r^5}$$

The two derivatives are identical, and continue to be so, even when the value of r is reduced to any assignable extent. In the same way we may show that just as

$$\frac{\partial\, \mathfrak{H}_y}{\partial\, \zeta} = \frac{\partial\, \mathfrak{H}_z}{\partial\, \eta}, \text{ so also } \frac{\partial\, \mathfrak{H}_z}{\partial\, \xi} = \frac{\partial\, \mathfrak{H}_x}{\partial\, \zeta}, \text{ and } \frac{\partial\, \mathfrak{H}_x}{\partial\, \eta} = \frac{\partial\, \mathfrak{H}_y}{\partial\, \xi}.$$

The equations (7) of § 39 are consequently fulfilled, so that everywhere without exception throughout the whole field the distribution of $\mathfrak{H}$ is lamellar; even at the place where the end-element itself is situated.

In the next place, let us form the derivatives of the components of magnetic force with respect to the co-ordinates of the point P, by differentiating the expressions given in § 46 with respect to ξ, η, ζ; thus we find

$$\frac{\partial \mathfrak{H}_x}{\partial \xi} = \frac{\mathfrak{J}_z\, dx\, dy\,[r^2 - 3\,(\xi - x)^2]}{r^5}$$

$$\frac{\partial \mathfrak{H}_y}{\partial \eta} = \frac{\mathfrak{J}_z\, dx\, dy\,[r^2 - 3\,(\eta - y)^2]}{r^5}$$

$$\frac{\partial \mathfrak{H}_z}{\partial \zeta} = \frac{\mathfrak{J}_z\, dx\, dy\,[r^2 - 3\,(\zeta - z)^2]}{r^5}$$

while for the sum of the right-hand terms of these equations we have the expression

$$\mathfrak{J}_z\, dx\, dy \frac{3\,r^2 - 3\,(\xi - x)^2 - 3\,(\eta - y)^2 - 3\,(\zeta - z)^2}{r^5}$$

Referring to (18), this expression is seen to be zero for finite values of r, while for infinitely small values of r it assumes the form $0/0$, and moreover it can be shown that in the latter case the expression does not vanish. Thus we have

$$(19) \quad . \quad . \quad \frac{\partial \mathfrak{H}_x}{\partial \xi} + \frac{\partial \mathfrak{H}_y}{\partial \eta} + \frac{\partial \mathfrak{H}_z}{\partial \zeta} = 0, \text{ when } r > 0$$

that is, the distribution of $\mathfrak{H}$ is solenoidal throughout the field, except in the place where the end-element itself is situated.

In accordance with the law of superposition VII (§ 43), we can now extend by summation to any number of elements the property just proved for a single end-element. Thus we arrive at the following theorem:

X. *The magnetic intensity due to a rigid magnet arbitrarily magnetised has everywhere a lamellar-solenoidal distribution, except in those places where end-elements are situated;* there the distribution is only lamellar.

§ 48. **Potential of a Rigid Magnet.**—In accordance with what we have just seen, the magnetic intensity due to a rigid magnet has everywhere a scalar potential, which satisfies Laplace's equation at every point, except in those places mentioned in Theorem X. This function is called the *magnetic potential* of the rigid magnet, and, as before (§ 45), we shall denote it by Υ.

Let us now turn our attention once more to the magnetic field due to the elementary parallelopiped. Let the face which has hitherto been considered (that which is shaded in fig. 11, p. 61) be distinguished by the number 1, the face immediately opposed to it by 4; and in the same way let the two remaining pairs of opposite faces be numbered 2 and 5, 3 and 6, respectively. We shall denote by $\delta\Upsilon_1$ the magnetic potential at the point P, due to the face 1, and from the general theory of potential it immediately follows that

$$\partial\Upsilon_1 = -\int_\infty^r \mathfrak{H}\,dr = -\int_\infty^r \frac{\mathfrak{J}_z\,dx\,dy}{r^2}\,dr = +\frac{\mathfrak{J}_z\,dx\,dy}{r}$$

Consider next the elementary face 4; let its distance from P be $r + dr$; its strength is $-\mathfrak{J}_z\,dx\,dy$. It produces at P a magnetic potential $\delta\Upsilon_4$, which is found exactly as in the case of 1; we have

$$\delta\Upsilon_4 = -\frac{\mathfrak{J}_z\,dx\,dy}{r + dr}$$

If we add together the two terms $\delta\Upsilon_1$ and $\delta\Upsilon_4$ of the potential, we obtain the magnetic potential due to the pair of faces (1 · 4), which we denote by $\delta\Upsilon_{1\cdot4}$, and which has accordingly the following value:—

$$\delta\Upsilon_{1\cdot4} = \delta\Upsilon_1 + \delta\Upsilon_4 = \mathfrak{J}_z\,dx\,dy\left(\frac{1}{r} - \frac{1}{r + dr}\right)$$

or, as it may also be written,

$$\delta\Upsilon_{1\cdot4} = -\mathfrak{J}_z\,dx\,dy\,d\frac{1}{r} = +\frac{\mathfrak{J}_2\,dx\,dy\,dr}{r^2}$$

On comparing this expression with that for $\delta\Upsilon_1$ or $\delta\Upsilon_4$, it will be seen that the magnetic potential of a pair of opposite end-elements is an infinitesimal of the third order, while that

of a single face belongs only to the second order of small quantities.

When we pass from the face 1 to the face 4 (compare fig. 11, p. 61), the Z-co-ordinate increases by $-dz$, and consequently

$$d\frac{1}{r} = -\frac{\partial \frac{1}{r}}{\partial z} dz$$

Substituting this value in the expression for $\delta \Upsilon_{1\cdot 4}$ we obtain

$$(20) \qquad . \qquad . \qquad \delta \Upsilon_{1\cdot 4} = \mathfrak{I}_z \frac{\partial \frac{1}{r}}{\partial z} dx\, dy\, dz$$

The pairs of opposite faces, 2 and 5 (strength $\pm \mathfrak{I}_x\, dy\, dz$) and 3 and 6 (strength $\mathfrak{I}_y\, dz\, dx$) are to be dealt with in exactly the same way. We obtain, then, by summation as the value of the magnetic potential at P, due to the whole parallelopiped,

$$\delta \Upsilon = \delta \Upsilon_{1\cdot 4} + \delta \Upsilon_{2\cdot 5} + \delta \Upsilon_{3\cdot 6}$$

If in this equation we insert the expression (20) and the corresponding expressions for $\delta \Upsilon_{2\cdot 5}$ and $\delta \Upsilon_{3\cdot 6}$, the result obtained is

$$\delta \Upsilon = \left(\mathfrak{I}_x \frac{\partial \frac{1}{r}}{\partial x} + \mathfrak{I}_y \frac{\partial \frac{1}{r}}{\partial y} + \mathfrak{I}_z \frac{\partial \frac{1}{r}}{\partial z} \right) dx\, dy\, dz$$

To find the potential at P due to the entire rigid magnet, $\delta \Upsilon$ must be integrated over the entire volume of the latter. This gives

$$\Upsilon = \iiint \left(\mathfrak{I}_x \frac{\partial \frac{1}{r}}{\partial x} + \mathfrak{I}_y \frac{\partial \frac{1}{r}}{\partial y} + \mathfrak{I}_z \frac{\partial \frac{1}{r}}{\partial z} \right) dx\, dy\, dz$$

If we now apply the principle of integration by parts to each term of the above integral separately, putting in the first term, for example,

$$\int \mathfrak{I}_x \frac{\partial \frac{1}{r}}{\partial x} dx = \frac{\mathfrak{I}_x}{r} - \int \frac{1}{r} \frac{\partial \mathfrak{I}_x}{\partial x} dx$$

and if we use these transformations for the complete triple integral, we obtain finally

$$(21) \qquad \Upsilon = \iint \frac{\mathfrak{J}_x\, dy\, dz + \mathfrak{J}_y\, dz\, dx + \mathfrak{J}_z\, dx\, dy}{r}$$
$$- \iiint \frac{1}{r} \left(\frac{\partial\, \mathfrak{J}_x}{\partial\, x} + \frac{\partial\, \mathfrak{J}_y}{\partial\, y} + \frac{\partial\, \mathfrak{J}_z}{\partial\, z} \right) dx\, dy\, dz$$

the double integral being extended over the bounding-surface of the rigid magnet, and the triple integral throughout the region contained within that surface.

§ 49. **Analogy with Gravitation-Potential.**—Let us consider more closely an element $d\,S'$ of the closed bounding-surface in question, and let $\mathfrak{N}$ be the normal drawn outward from the surface; then

$$d\,S' \cos\,(\mathfrak{N}, x) = dy\, dz$$
$$d\,S' \cos\,(\mathfrak{N}, y) = dz\, dx$$
$$d\,S' \cos\,(\mathfrak{N}, z) = dx\, dy$$

and if we insert these values in the first term of the expression (21) above, it becomes

$$\iint \frac{\mathfrak{J}_x \cos\,(\mathfrak{N}, x) + \mathfrak{J}_y \cos\,(\mathfrak{N}, y) + \mathfrak{J}_z \cos\,(\mathfrak{N}, z)}{r}\, d\,S'$$

or simply

$$\iint \frac{\mathfrak{J}_\nu}{r}\, d\,S'$$

Concerning the second term it should be observed that

$$(22) \qquad - \left(\frac{\partial\, \mathfrak{J}_x}{\partial\, x} + \frac{\partial\, \mathfrak{J}_y}{\partial\, y} + \frac{\partial\, \mathfrak{J}_z}{\partial\, z} \right) = \mathfrak{r}$$

is the convergence of the magnetisation (§ 35), and would therefore vanish if the distribution of $\mathfrak{J}$ happened to be solenoidal; we shall denote it by $\mathfrak{r}$. Accordingly, when we substitute these expressions in (21), the magnetic potential is given by the following fundamental equation:

$$(\mathrm{I}) \qquad \Upsilon = \iint \frac{\mathfrak{J}_\nu}{r}\, d\,S + \iiint \frac{\mathfrak{r}}{r}\, dx\, dy\, dz$$

This expression is applicable over the entire field, both within and without the rigid magnet considered; mathematically, it is strictly analogous to that for the gravitation-

potential when the masses are supposed to be distributed within the body in question with volume-density, $\mathfrak{r}$, while the surface of the body is supposed to be covered with a layer of matter whose surface-density has at each point the same numerical value as the component $\mathfrak{J}_\nu$ of the vector $\mathfrak{J}$. If for a moment, then, we assume the existence of magnetic fluids (§ 27)—an assumption which was formerly general—the potential of this fictitious distribution will be identical with that which we have deduced from our standpoint.

If we recall to mind the method by which our subject has been developed, the physical conception of magnetisation was first set forth on the basis of experimental investigations. Magnetic intensity exerted at a distance was then considered as due to magnetically active end-elements, whose strength was measured by the product of their area into the component of magnetisation in the direction of their normal. But provided we do not ascribe a real physical existence to the magnetic fluids, there can be no objection to our making use of the purely mathematical analogy with the laws of gravitation-potential[1] whenever we find, in our present subject, that we can thus arrive more shortly at new results.

§ 50. **Local Variations of Magnetic Strength as Centres of Action.**—We shall now make use of the well-known equation of Poisson:

$$(23) \quad . \quad . \quad . \quad . \quad \nabla^2 \Upsilon = -4\pi\mathfrak{r}$$

by which we have to replace Laplace's equation wherever there is gravitating matter of volume-density, $\mathfrak{r}$. Performing the operation indicated by ∇^2 (see § 41), and substituting for $\mathfrak{r}$ its value from (22), we have, in the present case,

$$(24) \quad -\left(\frac{\partial \mathfrak{H}_x}{\partial x} + \frac{\partial \mathfrak{H}_y}{\partial y} + \frac{\partial \mathfrak{H}_z}{\partial z}\right) = 4\pi\left(\frac{\partial \mathfrak{J}_x}{\partial x} + \frac{\partial \mathfrak{J}_y}{\partial y} + \frac{\partial \mathfrak{J}_z}{\partial z}\right)$$

Thus, whenever the convergence of the magnetisation is finite at a given point, the convergence of the magnetic intensity at that point is also finite, the latter being equal to -4π times the

[1] See Lejeune-Dirichlet, *Vorlesungen über die im umgekehrten Verhältniss des Quadrats der Entfernung wirkenden Kräfte*, Leipzig, 1876; Clausius, *Die Potentialfunktion und das Potential*; Thomson and Tait, *Nat. Philosophy*, vol. 2.

former. This can be otherwise expressed by saying that a solenoidal distribution of magnetisation constitutes the necessary and sufficient condition for a solenoidal distribution of the resulting magnetic intensity.

In the course of our enquiry we have already introduced the conception of magnetic distributions of end-elements as centres of action at a distance, and we are now in a position to define more exactly the part played by such end-elements. From equation (I) it follows that the magnetic potential due to a rigid magnet arises partly from its bounding-surface, and, moreover, it is easy to see that at the surface the *tubes of magnetisation* terminate abruptly, thus determining for each element of surface, $d\,S$, the magnetic strength $\mathfrak{J}_\nu\, d\,S$, and giving rise to the first term on the right hand of equation (I). It may be remarked in passing that in most cases the portion of Υ corresponding to this term largely preponderates.

The second term of the magnetic potential arises from the condition of things within the magnet. For convenience of analytical treatment we have supposed the whole volume to be divided up into elementary parallelopipeds, whose terminal faces, however, have of course a purely fictitious existence; moreover, the effects due to the adjacent faces of two contiguous parallelopipeds will, in great part, neutralise one another, since they are of opposite signs. If the magnetisation happens to be solenoidally distributed at the place considered, this compensation will be complete. On the other hand, if this is not the case, the second term of equation (I), which is due to the distribution within the magnet, may be considered as analogous to the first, the expression

$$\mathfrak{r}\, dx\, dy\, dz = -\left(\frac{\partial\, \mathfrak{J}_x}{\partial\, x} + \frac{\partial\, \mathfrak{J}_y}{\partial\, y} + \frac{\partial\, \mathfrak{J}_z}{\partial\, z}\right) dx\, dy\, dz$$

[compare (22)] taking the place of the end-element of magnetic strength $\mathfrak{J}_\nu d\,S$, which occurs in the first term. The above expression now appears as the convergence of the magnetisation in the neighbourhood of the volume-element considered, multiplied by the volume of the element. The two kinds of centres of action at a distance (surface-elements and volume-elements) may now finally be regarded from a common

standpoint. Whenever the convergence becomes finite, the distribution of the magnetisation ceases to be solenoidal, and consequently the strength of the tubes of magnetisation (magnetisation × area of normal section) is no longer constant (§ 37). It is, moreover, immediately evident that, as we pass outwards through the surface bounding the magnet, the strength of the tubes of magnetisation suddenly falls to zero. We have, therefore, quite generally :—

XI. *Centres of magnetic action occur only in those places where the strength of the tubes of magnetisation is variable from point to point, whether these variations be abrupt or continuous.*[1]

Thus, all magnetic actions at a distance may be referred to centres, whose existence is conditioned by the presence of local variations of magnetic intensity.

§ 51. **Intensity and Induction within Ferromagnetic Media.** The expression (I) for the magnetic potential holds good for all points, whether lying within or without the rigid magnet. Its differential coefficients with respect to the co-ordinates, each with its sign reversed, give the components of magnetic intensity. The question now arises what is the physical meaning of this latter vector at a point within the substance of the magnet, since it was only originally introduced as a measure of the magnetic condition in a region either unoccupied or filled with magnetically indifferent material (§ 4). To gain a clearer insight into this, let us suppose the magnet to be hollowed out at the point in question, and consider the magnetic condition in the cavity thus formed. This depends, as may be shown, on the shape of the cavity, and in several simple cases its value may be calculated. Since these calculations themselves are not of a fundamental nature, we shall content ourselves with a statement of results, referring for the proof to more exhaustive treatises.[2]

[1] This was formerly expressed by saying that at such places 'free magnetism' occurred. This term, which is not very happily chosen, and far from easy to define with exactness, is frequently to be met with, even at the present day; moreover, it is sometimes used to signify merely the distribution of 'magnetism' on the *surface* of the magnet.

[2] Compare Maxwell, *Treatise*, vol. 2, chap. ii.; Mascart and Joubert, *Electricity and Magnetism*, vol. 1, §§ 321–324.

First let the cavity be in the form of a thin tube or *crevass* parallel to the direction of magnetisation; then the magnetic intensity $\mathfrak{H}'$ within the cavity is directly deducible from the magnetic potential, whose value was found above; it is defined as *the magnetic intensity* in a ferromagnetic medium. Hence we write

$$(25) \quad . \quad \mathfrak{H}'_x = -\frac{\partial \Upsilon}{\partial x}, \quad \mathfrak{H}'_y = -\frac{\partial \Upsilon}{\partial y}, \quad \mathfrak{H}'_z = \frac{\partial \Upsilon}{\partial z}$$ [1]

Secondly, suppose the cavity to have the form of a thin flat crevass, whose plane is perpendicular to the lines of magnetisation; the magnetic intensity within the cavity is now no longer the same. If it be denoted by $\mathfrak{B}'$, it may be shown that

$$(26) \quad . \quad . \quad . \quad \begin{cases} \mathfrak{B}'_x = \mathfrak{H}'_x + 4\pi\mathfrak{J}_x \\ \mathfrak{B}'_y = \mathfrak{H}'_y + 4\pi\mathfrak{J}_y \\ \mathfrak{B}'_z = \mathfrak{H}'_z + 4\pi\mathfrak{J}_z \end{cases}$$

These results may be more simply expressed by the single equation

$$(27) \quad . \quad . \quad . \quad \mathfrak{B}' = \mathfrak{H}' + 4\pi\mathfrak{J}$$

but this, we must remember, is a so-called vector-equation; it is only in the special case when $\mathfrak{B}'$, $\mathfrak{H}'$ and $\mathfrak{J}$ have the same direction at the point considered that the equation interpreted in the ordinary (scalar) manner holds good for these vectors themselves, and not merely for their components; in the former sense the equation has already been introduced (§ 11).

The vector $\mathfrak{B}'$ thus defined is named *the magnetic induction* in a ferromagnetic medium, since it is evidently identical with the quantity already designated by this name (§ 10), though its definition was then based upon induction-experiments. We have directed our attention exclusively to the field due to the rigid magnet itself; if, in addition, there is a field arising from independent causes, the magnetic intensity due to this field must be added (vectorially) both to $\mathfrak{H}'$ and to $\mathfrak{B}'$, so that equation (27) continues to hold good as before.

§ 52. **Magnetic Induction is distributed Solenoidally.**—The fundamental property of the magnetic induction as just de-

[1] Here and henceforth, the accents attached to the symbols signify that we are dealing with the values of the quantities represented at points within the ferromagnetic substance.

fined is this; that throughout the whole field without exception, and under all circumstances, its distribution is solenoidal, although in general not lamellar. We have, therefore, to prove that it satisfies the two forms of the equation of continuity (§ 37): 1st, that which is applicable throughout any region where the vector itself is continuous, and, 2nd, the boundary equation at those surfaces where the vector may abruptly change its value; for these equations constitute the necessary and sufficient condition for a solenoidal distribution.

1. From the relations (26), which we may consider as the definition of magnetic induction, it follows that

$$\frac{\partial \mathfrak{B}'_x}{\partial x} + \frac{\partial \mathfrak{B}'_y}{\partial y} + \frac{\partial \mathfrak{B}'_z}{\partial z}$$

$$= \frac{\partial \mathfrak{H}'_x}{\partial x} + \frac{\partial \mathfrak{H}'_y}{\partial y} + \frac{\partial \mathfrak{H}'_z}{\partial z} + 4\pi \left(\frac{\partial \mathfrak{J}_x}{\partial x} + \frac{\partial \mathfrak{J}_y}{\partial y} + \frac{\partial \mathfrak{J}_z}{\partial z}\right)$$

But by equation (24) the right-hand member of this last equation is always zero; for, if the vector $4\pi\mathfrak{J}$ has a convergence different from zero, this will be cancelled by the equal and opposite convergence of the vector $\mathfrak{H}'$, so that the vector-sum of $\mathfrak{H}'$ and $4\pi\mathfrak{J}$ can have no convergence.

Outside the ferromagnetic body we have further $\mathfrak{J} = 0$, so that $\mathfrak{B}$ and $\mathfrak{H}$ are identical at every point; but in this case we have (§ 47)

$$\frac{\partial \mathfrak{H}_x}{\partial x} + \frac{\partial \mathfrak{H}_y}{\partial y} + \frac{\partial \mathfrak{H}_z}{\partial z} = 0$$

so that the equation of continuity

$$\frac{\partial \mathfrak{B}_x}{\partial x} + \frac{\partial \mathfrak{B}_y}{\partial y} + \frac{\partial \mathfrak{B}_z}{\partial z} = 0$$

is satisfied at all points of the field, whether we consider the ferromagnetic body itself or the space outside it.

2. The only surface of discontinuity which we have to consider is the bounding-surface of the magnet itself. For the sake of shortness, let us make use of the analogy of the known theorem in gravitation, that, at a surface on which matter is spread with surface-density $\mathfrak{s}$, the space-derivative of the gravitation-potential, reckoned along the normal, changes

abruptly by $4\pi\mathfrak{s}$. Applying this result to the present case, we have

$$\mathfrak{H}_\nu = \mathfrak{H}'_\nu + 4\pi\mathfrak{J}_\nu$$

where $\mathfrak{H}_\nu$ and $\mathfrak{H}'_\nu$ represent the components of magnetic force measured along the normal to the surface, just without and just within the magnet respectively. But by the definition already given, we have, on the one hand, within the magnet [equation (27)]

$$\mathfrak{B}'_\nu = \mathfrak{H}'_\nu + 4\pi\mathfrak{J}_\nu$$

and, on the other hand, outside the magnet,

$$\mathfrak{B}_\nu = \mathfrak{H}_\nu$$

Combining the last three equations we obtain

$$(28) \quad . \quad . \quad . \quad . \qquad \mathfrak{B}_\nu = \mathfrak{B}'_\nu$$

From this it follows that the boundary equation of continuity is satisfied at the surface of the magnet, so that the solenoidal character of the distribution of magnetic induction throughout the entire field is established. We shall recur to this theorem in connection with the phenomena of magnetic electric induction (§ 61); to the consideration of these we now proceed, the outlines of the theory of rigid magnets having been explained in the present chapter.

CHAPTER IV

OUTLINES OF THE THEORY OF MAGNETIC INDUCTION

§ 53. **Magnetic Intensity due to Magnetisation and to External Causes.**—In the last chapter, when we were dealing with the properties of rigid magnets, we assumed their magnetisation as given, without making any further enquiry into its cause.

We have already seen (§ 8) that ferromagnetic bodies only become magnetised when they are subject to the inductive influence of a magnetic field, though after the direct influence of the field has ceased, there remains the lagging effect to which the name of hysteresis is given. Up to the present time, no complete theory of ferromagnetic induction has been formulated, wherein hysteresis is taken into account; and indeed the investigation would be beset with difficulties. We accordingly make the restriction already explained above, that hysteresis is to be entirely neglected, *the magnetisation being therefore a function of the resultant magnetic intensity alone.*

Now we have to remember, in the first place, that this magnetic intensity may arise from external influences; for example, from such rigid magnets as may be in the neighbourhood of the body, or from electric currents flowing in neighbouring conductors; and, in the second place, we have to consider that part of the magnetic intensity which is due to the magnetisation of the body itself, and whose properties were investigated in the preceding chapter. The latter portion we shall distinguish by affixing the suffix *i* (internal), denoting it therefore by $\mathfrak{H}_i$. On the other hand, the remaining portion of the magnetic intensity, arising from any other causes whatever, will be distinguished by the suffix *e* (external). The resultant sum of the two parts (understanding the word in its vectorial sense) determines the intensity

of the induced magnetisation, and will be characterised by the use of the suffix t (total).[1] We have accordingly

$$\mathfrak{H}_{tx} = \mathfrak{H}_{ex} + \mathfrak{H}_{ix}, \quad \mathfrak{H}_{ty} = \mathfrak{H}_{ey} + \mathfrak{H}_{iy}, \quad \mathfrak{H}_{tz} = \mathfrak{H}_{ez} + \mathfrak{H}_{iz}$$

or, combining these results in a vector equation,

$$(1) \qquad \mathfrak{H}_t = \mathfrak{H}_e + \mathfrak{H}_i$$

In the same manner we shall attach suffixes to Υ (the magnetic potential) and to $\mathfrak{B}$ (the magnetic induction), so as to distinguish among the various causes from which they may arise. This classification is condemned by some authors as artificial; but in the further development of our subject exact mathematical treatment would be impossible without this classification.

§ 54. **Kirchhoff's Assumptions.**—Consider the case of a body of homogeneous isotropic ferromagnetic material,[2] through which no electric current is flowing, the shape of the body being entirely unrestricted. Since, in general, the body will not be in the form of a ring or other endless shape (in which case perchance there might be no magnetically active end-elements), its own magnetisation will give rise to a field $\mathfrak{H}_i$, which must be added to the previously existing field $\mathfrak{H}_e$ due to external causes. This latter vector may be distributed in any manner, provided the distribution is such as to satisfy the general laws enunciated in the last chapter.

Accordingly, whether $\mathfrak{H}_e$ arises from electric currents or from external magnets, its distribution throughout the space occupied by the body in question must be lamellar and solenoidal.[3] If we take the line-integral of $\mathfrak{H}_e$ along a closed path which embraces q times a conductor conveying a current I, the value

[1] When we find it necessary to use more than one suffix, those which indicate the source of the magnetic force will be placed first; those relating to the direction coming after (§ 34).

[2] Homogeneity and isotropy imply, as we know, that the properties of the body are independent of the particular part considered, and independent also of direction in the body. The theory of magnetic induction in aelotropic media, not belonging to the class of ferromagnetic substances, has indeed been developed mathematically, and tested experimentally, but would not be applicable to ferromagnetic substances of variable susceptibility such as those which we here consider. For our purpose the restriction to isotropy is of no importance, since the ferromagnetic metals in common use may in general be considered as isotropic.

[3] This follows from § 44, Theorem VIII § 45 and Theorem X § 47.

obtained is $4 \pi q I$, which evidently vanishes in the case in which the path of integration embraces no current whatever (Theorem IX § 45).

If the body in question is introduced into this externally generated field, or if the field is suddenly produced in the space already occupied by the body, the distribution of magnetisation determined by the special conditions of the case will be acquired within a very short interval of time. We have already shown that the magnetisation must be determined in value and direction by the *total magnetic intensity* $\mathfrak{H}_t$, and a further question now arises as to how these two quantities are related to one another.

In the old theory of magnetic induction, which was first developed by Poisson, and was based on the assumption of two magnetic fluids (§ 27), and which was afterwards further elaborated by Lord Kelvin, F. Neumann, Maxwell, and others, these two vectors were assumed to be proportional to one another in value, and in the same direction. But the assumption of proportionality is now known to have been unjustifiable; notwithstanding which it is often retained, even in recent literature, in order to facilitate calculation, though experiment has long ago shown it to be untenable. Kirchhoff, therefore, in 1853 made two new assumptions,[1] which hold good provided that we need not take hysteresis into account. We now proceed to give an account of these assumptions, which form the starting point of theory.

In the first place, it follows from considerations of symmetry that the direction of the magnetisation at each point must be the same as that of the vector $\mathfrak{H}_t$, since this is the sole cause from which the vector arises. In fact, it is impossible to assign any reason for supposing that the direction of $\mathfrak{J}$ would be different from that of $\mathfrak{H}_t$, for in an isotropic medium there is nothing to cause a deviation in one direction rather than in another.

In the second place, the numerical value of $\mathfrak{J}$ must depend only on that of $\mathfrak{H}_t$, without, however, being proportional to it; that is—

(2) . . . $\mathfrak{J} = \text{funct.}\ (\mathfrak{H}_t)$

[1] Kirchhoff, *Crelle's Journal*, vol. 48, p. 370, 1853; *Gesammelte Abhandlungen*, p. 217.

This function must be that which is represented by the normal curve of magnetisation characteristic of the substance in question (§ 13); and this applies to the case of a body of endless shape, where, owing to the absence of magnetically active ends, $\mathfrak{H}_i = 0$, and we have $\mathfrak{H}_t = \mathfrak{H}_e$. We found that the curve possessed this fundamental property: that as $\mathfrak{H}_t$ increased indefinitely, $\mathfrak{I}$ approached asymptotically a limiting maximum value. If we introduce the value of the susceptibility, that is, the ratio of the magnetisation to the inducing magnetic intensity (§ 14), denoting this quantity by $\kappa\ (\mathfrak{H}_t)$, so as to indicate that it also is a function of $\mathfrak{H}_t$ alone, we may equally well write

$$\mathfrak{I} = \kappa\ (\mathfrak{H}_t) . \mathfrak{H}_t$$

and since $\mathfrak{I}$ and $\mathfrak{H}_t$ are in the same direction, we shall have

(3) $\quad \mathfrak{I}_x = \kappa\ (\mathfrak{H}_t)\ \mathfrak{H}_{tx}, \ \mathfrak{I}_y = \kappa\ (\mathfrak{H}_t)\ \mathfrak{H}_{ty}, \ \mathfrak{I}_z = \kappa\ (\mathfrak{H}_t)\ \mathfrak{H}_{tz}$

The fundamental relation connecting the three vectors $\mathfrak{B}$, $\mathfrak{H}$, and $\mathfrak{I}$ [§ 51, equation (27)] becomes in the present case

(4) $\quad . \quad . \quad . \quad \mathfrak{B}_t = \mathfrak{H}_t + 4\,\pi\,\mathfrak{I}$

From this it follows that the *resultant induction* $\mathfrak{B}_t$ must be in the same direction as its two terms $\mathfrak{H}_t$ and $4\,\pi\,\mathfrak{I}$, which are themselves coincident in direction. To express this formally, we must introduce the permeability $\mu\ (\mathfrak{H}_t)$, that is, the ratio of the induction to the magnetic intensity (§ 14). Thus we obtain

$$\mathfrak{B}_t = \mu\ (\mathfrak{H}_t) . \mathfrak{H}_t$$

And since $\mathfrak{B}_t$ and $\mathfrak{H}_t$ are in the same direction, this is equivalent to

(5) $\quad \mathfrak{B}_x = \mu\ (\mathfrak{H}_t)\ \mathfrak{H}_{tx}, \quad \mathfrak{B}_y = \mu\ (\mathfrak{H}_t)\ \mathfrak{H}_{ty}, \quad \mathfrak{B}_z = \mu\ (\mathfrak{H}_t)\ \mathfrak{H}_{tz}$

§ 55. **Line-integral of the Demagnetising Intensity.**—That part of the magnetic intensity which arises from the magnetisation of the body itself we shall denote by $\mathfrak{H}_i$ in the space outside the body; in accordance with § 47, its distribution is there lamellar and solenoidal. When we are dealing with the interior of the body, we shall further distinguish this quantity by adding an accent, $\mathfrak{H}_i'$ being called the *demagnetising intensity*, since in general its tendency is to oppose the external field $\mathfrak{H}_e$, and so to diminish the magnetisation of the body. Now, within the sub-

stance of the body the distribution of $\mathfrak{H}'_i$, though still lamellar, is no longer solenoidal, since the convergence of the magnetisation has, in general, a finite value, which we have shown (§ 50) to be equal to -4π times the convergence of the vector $\mathfrak{H}'_i$. Now, since $\mathfrak{H}_i$ (or $\mathfrak{H}'_i$, as the case may be) has everywhere a lamellar distribution, the magnetic intensity due to the magnetisation of the body must always be derivable from a magnetic potential Υ_i, and in accordance with what we have already proved (§ 40), the line-integral of $\mathfrak{H}_i$ or $\mathfrak{H}'_i$ taken round any closed curve must vanish.

Let us consider more particularly such a closed path of integration, of which part lies within the ferromagnetic substance, and part in the region outside, the curve being traversed in the direction indicated by the arrow (fig. 12).

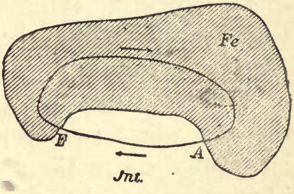

FIG. 12

For the portion $\overline{EA}$ of the path which lies within the ferromagnetic substance, the line-integral of the demagnetising intensity is

$$\int_E^A \mathfrak{H}'_{iL}\, d\, L\, ;$$

where $\mathfrak{H}'_{iL}$ denotes, as usual, the component of the vector in question in the direction of the tangent to the curve L, so that

$$\mathfrak{H}'_{iL} = \mathfrak{H}'_i \cos\,(\mathfrak{H}'_i,\, L)$$

On the other hand, the portion $\overline{AE}$ of the path which lies without the ferromagnetic substance furnishes the line-integral

$$\int_A^E \mathfrak{H}_{iL}\, dL = {}_A^E\Upsilon_i$$

which is equal to the increment experienced by the magnetic potential due to magnetisation as we pass from the point where the path of integration leaves the body to that where it enters again. Now, we have already seen that[1]

$$\int_A^A \mathfrak{H}_{iL}\, dL = 0 = \int_A^E \mathfrak{H}_{iL}\, d\, L + \int_E^A \mathfrak{H}'_{iL}\, dL$$

[1] Since, in integrating along a closed curve, we have the initial and final extremities coincident, we shall in the sequel denote such an integral by the symbol ${}_A^A\!\int$.

so that

$$(6) \qquad - {}_{A}^{E}\Upsilon_i = \int_E^A \mathfrak{H}'_{iL}\, dL = {}_{E}^{A}\Upsilon_i$$

This equation shows that—

I. *Between any two points, E and A, of the bounding-surface of a ferromagnetic body, the difference of value of the magnetic potential due to its own magnetisation is numerically equal to the line-integral of demagnetising intensity, reckoned along a path from E to A, which lies within the body.*[1]

The applicability of this equation will become apparent later on, when we have to effect the solution of special problems. It will be seen that, in virtue of Theorem I, the difference of magnetic potential ${}_{A}^{E}\Upsilon_i$, which can be calculated or measured in the accessible external medium, may be made to furnish a basis for calculating the demagnetising influence within the ferromagnetic body.

It is easy to see that the proposition may also be extended to the case where two or more separate portions of the path of integration lie in the surrounding medium, so that the curve in question cuts the bounding surface in more than two points, though, of course, always in an even number. If the points where the curve passes *out of* the ferromagnetic body, taken in order, be denoted by $A_1, A_2, \ldots . A_n$, the points where it enters the body being called $E_1, E_2, \ldots . E_n$, we shall have, instead of equation (6), the following, which may be proved analytically in the same manner :—

$$(6a) \qquad \begin{aligned} &- {}_{A_1}^{E_1}\Upsilon_i - {}_{A_2}^{E_2}\Upsilon_i - \ldots\ldots - {}_{A_n}^{E_n}\Upsilon_i \\ &= \int_{E_1}^{A_2} \mathfrak{H}'_{iL}\, dL + \int_{E_2}^{A_3} \mathfrak{H}'_{iL}\, dL + \ldots . + \int_{E_n}^{A_1} \mathfrak{H}'_{iL}\, dL \end{aligned}$$

By sketching a figure to illustrate this more general case, the reader may immediately satisfy himself of the truth of equation (6a). Thus, we see that :—

I.A. *If any circuit be drawn, of which some portions lie within the ferromagnetic body and some portions in the surrounding medium, the sum of the increments of the body's own magnetic potential corresponding to the latter portions is numerically equal*

[1] This theorem was given by the author in *Wied. Ann.*, vol. **46**, p. 489, 1892. Throughout the demonstration, however, $\mathfrak{H}_i$ was wrongly printed in the place of $\mathfrak{H}_{iL}$.

to the sum of the line-integrals of demagnetising intensity reckoned along the former portions.

§ 56. **Properties of Resultant Magnetic Intensity.**—In accordance with the law of superposition (VII, § 43), $\mathfrak{H}_t$, being the sum of two lamellarly distributed vectors $\mathfrak{H}_e$ and $\mathfrak{H}_i$, has itself a distribution which is everywhere lamellar, though not necessarily solenoidal. Again, the line-integral of the resultant magnetic intensity, reckoned along a closed curve (indicated by $\int_A^A$), becomes

$$\int_A^A \mathfrak{H}_{tL}\, dL = \int_A^A \mathfrak{H}_{eL}\, dL + 0 = 4\,\pi\, q\, I$$

in the case where the curve embraces q times a conductor conveying a current I.

The expression for the magnetic potential Υ_i due to the body itself has already been compared with that for the gravitation-potential, and from this it follows that at surfaces of discontinuity the magnetic potential must remain continuous, since this property is known to appertain to the gravitation-potential. Accordingly, at any point of the surface which separates the ferromagnetic from the external medium, the space-derivative of Υ_i in any direction lying in the tangent-plane must have the same value within and without the surface; that is,

$$\mathfrak{H}_{i_\tau} = \mathfrak{H}'_{i_\tau}$$

On the other hand, we have already shown (§ 52) that, as in the case of gravitation-potential, so also in our case, the space-derivative along the normal at a point of the bounding-surface, experiences an abrupt change of value, equal to $4\,\pi\,\mathfrak{J}_\nu$. Thus we have

$$\mathfrak{H}'_{i\nu} = \mathfrak{H}'_{i\nu} + 4\,\pi\,\mathfrak{J}_\nu$$

And, further, since there is nothing which could cause the vector $\mathfrak{H}_e$ to exhibit any special peculiarities at the bounding-surface of the body, the vector-sum $\mathfrak{H}_t = \mathfrak{H}_e + \mathfrak{H}_i$ must there show the same discontinuity as its second term alone. Thus,

(7) . . . . $\mathfrak{H}_{t_\tau} = \mathfrak{H}'_{t_\tau}$

While, on the other hand,

(8) . . . $\mathfrak{H}_{t\nu} = \mathfrak{H}'_{t\nu} + 4\,\pi\,\mathfrak{J}_\nu$

From equations (7) and (8) we see that :—

II. *The tangential component of the resultant magnetic intensity is continuous at the bounding-surface of the body; while the normal component is discontinuous at the surface wherever its value (and consequently also that of the normal component of magnetisation) is finite.*[1]

§ 57. **Properties of Magnetisation.**—In accordance with our first proposition, the vector $\mathfrak{I}$ is at each point coincident in direction with $\mathfrak{H}_t$; so that the lines of magnetisation everywhere coincide with the lines of resultant magnetic intensity. Since the last-mentioned vector is lamellarly distributed, and is derivable therefore from a potential Υ_t, the corresponding lines of magnetic intensity cut the system of equipotential surfaces (Υ_t=const.) orthogonally; and the numerical value of $\mathfrak{H}_t$ is inversely proportional to the distance between two consecutive equipotential surfaces (§ 39). Evidently, therefore, the lines of magnetisation must cut the same system of equipotential surfaces orthogonally, though in general the same relation does not hold good between the numerical value of this vector and the distance between neighbouring surfaces of the system. The existence of a system of surfaces, which are everywhere orthogonal to the direction of the magnetisation, shows that the distribution of this vector must at least be complex-lamellar (§ 38). This can also be proved analytically by taking as our starting-point the relation previously established

$$\mathfrak{I} = \kappa\,(\mathfrak{H}_t).\,\mathfrak{H}_t$$

and substituting this value in the equation which conditions the complex-lamellar distribution of $\mathfrak{I}$ [§ 38, equation (6)]. Thus we obtain

$$\mathfrak{I}_x\left(\frac{\partial\,\mathfrak{I}_y}{\partial\,z} - \frac{\partial\,\mathfrak{I}_z}{\partial\,y}\right) = \kappa\,\mathfrak{H}_{tx}\left(\kappa\,\frac{\partial\,\mathfrak{H}_{ty}}{\partial\,z} + \mathfrak{H}_{ty}\frac{\partial\,\kappa}{\partial\,z} - \kappa\,\frac{\partial\,\mathfrak{H}_{tz}}{\partial\,y} - \mathfrak{H}_{tz}\frac{\partial\,\kappa}{\partial\,y}\right)$$

$$\mathfrak{I}_y\left(\frac{\partial\,\mathfrak{I}_z}{\partial\,x} - \frac{\partial\,\mathfrak{I}_x}{\partial\,z}\right) = \kappa\,\mathfrak{H}_{ty}\left(\kappa\,\frac{\partial\,\mathfrak{H}_{tz}}{\partial\,x} + \mathfrak{H}_{tz}\frac{\partial\,\kappa}{\partial\,x} - \kappa\,\frac{\partial\,\mathfrak{H}_{tx}}{\partial\,z} - \mathfrak{H}_{tx}\frac{\partial\,\kappa}{\partial\,z}\right)$$

$$\mathfrak{I}_z\left(\frac{\partial\,\mathfrak{I}_x}{\partial\,y} - \frac{\partial\,\mathfrak{I}_y}{\partial\,x}\right) = \kappa\,\mathfrak{H}_{tz}\left(\kappa\,\frac{\partial\,\mathfrak{H}_{tx}}{\partial\,y} + \mathfrak{H}_{tx}\frac{\partial\,\kappa}{\partial\,y} - \kappa\,\frac{\partial\,\mathfrak{H}_{ty}}{\partial\,x} - \mathfrak{H}_{ty}\frac{\partial\,\kappa}{\partial\,x}\right)$$

[1] This theorem constitutes the theoretical foundation for the 'Isthmus method' introduced by Ewing and Low for determining the magnetisation in very intense fields. See Ewing, *Magnetic Induction.*

In each of the three bracketed expressions on the right, the first and third terms destroy one another, because the distribution of $\mathfrak{H}_t$ is lamellar; and since, on adding the three equations together, the sum of the remaining terms on the right hand vanishes identically, the sum of the terms on the left hand must also vanish; that is, the equation conditioning a complex-lamellar distribution of $\mathfrak{J}$ is likewise identically satisfied.[1] The quantity which appears as an integrating divisor is the susceptibility $\kappa(\mathfrak{H}_t)$, as we may easily verify by remarking that the expression

$$\mathfrak{J}_x\,dx + \mathfrak{J}_y\,dy + \mathfrak{J}_z\,dz$$

becomes immediately integrable on division by $\kappa(\mathfrak{H}_t)$. We have [§ 54, equation (3)]:

$$\frac{\mathfrak{J}_x}{\kappa(\mathfrak{H}_t)} = \mathfrak{H}_{tx}, \qquad \frac{\mathfrak{J}_y}{\kappa(\mathfrak{H}_t)} = \mathfrak{H}_{ty}, \qquad \frac{\mathfrak{J}_z}{\kappa(\mathfrak{H}_t)} = \mathfrak{H}_{tz}$$

so that the above differential expression assumes on division by $\kappa(\mathfrak{H}_t)$ the form

$$\mathfrak{H}_{tx}\,dx + \mathfrak{H}_{ty}\,dy + \mathfrak{H}_{tz}\,dz = -\,d\Upsilon_t$$

which is an exact differential.

We must not conclude without explaining an important property of the vector $\mathfrak{J}$. If we suppose the intensity $\mathfrak{H}_e$ of the external field to increase without limit, the second term on the right hand of the vector-equation

$$\mathfrak{H}_t = \mathfrak{H}_e + \mathfrak{H}_i$$

becomes continually smaller in comparison with the first. Thus $\mathfrak{H}_t$, as regards both value and direction, comes to depend almost exclusively on $\mathfrak{H}_e$; and the same must consequently hold good for the direction of the magnetisation, though the value of magnetisation cannot exceed the maximum or saturation-value. From this follows the *law of saturation* enunciated by Kirchhoff,[2] which may be expressed as follows:

[1] According to the earlier theory already noticed, the distribution of the magnetisation would have to be lamellar and solenoidal. This proposition, which formerly was frequently adduced, the author has replaced by the considerations explained in the text (*Wied. Ann.* vol. **46**, p. 491, 1892).

[2] Kirchhoff, *Gesammelte Abhandlungen*, p. 223.

III. *A ferromagnetic body of any shape is placed in an external magnetic field of force, distributed in any manner. If, then, the intensity of the external field is made to increase indefinitely, the direction of magnetisation at every point of the body will approach that of the external field, and its value will approach the maximum value attainable in the substance considered.*

§ 58. **Properties of the Resultant Magnetic Induction**—Since throughout the space occupied by the body under consideration the magnetic intensity $\mathfrak{H}_e$ arising from external causes is identical with the corresponding magnetic induction $\mathfrak{B}_e$ ($\mathfrak{H}_e$ being due to the action of external bodies, whose magnetisation at the place considered is necessarily zero), the distribution of $\mathfrak{B}_e$ must, like that of $\mathfrak{H}_e$, be lamellar and solenoidal. But by a fundamental theorem which has already been fully explained, $\mathfrak{B}_i$ is solenoidally distributed under all circumstances, whatever may be the distribution of $\mathfrak{I}$. Hence, in accordance with the law of superposition (VII § 43) the vector-sum $\mathfrak{B}_t = \mathfrak{B}_e + \mathfrak{B}_i$ is likewise distributed solenoidally.

The distribution of the vector $\mathfrak{B}_t$ is, moreover, complex-lamellar; the methods of proof already employed in the case of $\mathfrak{I}$ being applicable in this case also; one method depending on purely geometrical considerations, and the other on analytical processes, though the proof contained in the foregoing section must be suitably modified by the introduction of the permeability μ in place of the susceptibility κ, the former now appearing in place of the latter as the integrating divisor. Thus the resultant induction has been shown to have a complex-lamellar solenoidal distribution; so that in general this vector is not derivable from a scalar potential.

As we found for the case of $\mathfrak{H}_t$, so we may show that the boundary conditions to be satisfied by $\mathfrak{B}_t$ at the surface separating the ferromagnetic body from the surrounding medium are identical with those for $\mathfrak{B}_i$. We have already seen [§ 52, equation (28)] that this latter vector satisfies the surface-equation of continuity

$$\mathfrak{B}_{i\nu} = \mathfrak{B}'_{i\nu}$$

Hence also

(9) . . . . . $\mathfrak{B}_{t\nu} - \mathfrak{B}'_{\nu}$

On the other hand, we found [§ 56, equation (7)] that

$$\mathfrak{H}_{i_\tau} = \mathfrak{H}'_{i_\tau}$$

But at points of the ferromagnetic substance just within the bounding-surface

$$\mathfrak{B}'_{i_\tau} = \mathfrak{H}'_{i_\tau} + 4\,\pi\,\mathfrak{J}_\tau$$

while in the magnetically indifferent space outside

$$\mathfrak{B}_{i_\tau} = \mathfrak{H}_{i_\tau}$$

and from the last three equations we obtain

$$\mathfrak{B}'_{i_\tau} = \mathfrak{B}_{i_\tau} + 4\,\pi\,\mathfrak{J}_\tau \tag{10}$$

The following proposition is contained in equations (9) and (10):—

IV. *The normal component of the resultant induction is continuous at the bounding-surface of the body, but the tangential component is discontinuous at the surface whenever its value (and consequently that of the normal component of magnetisation) is finite.*[1]

Thus, the boundary conditions to be satisfied by the resultant induction are just the reverse of those which we found for the resultant magnetic force (Theorem II, § 56).

§ 59. **Practical Approximation.**—In a large number of cases which arise in practice the value of $\mathfrak{H}_i$, as already mentioned, (§ 11), does not exceed a few hundred C.G.S. units, and for all known ferromagnetic substances the relation

$$\mathfrak{J} = \frac{1}{4\,\pi}\,\mathfrak{B}_i$$

is then very approximately true; so that in such cases the distribution of the magnetisation may be regarded as sensibly solenoidal, since this vector and the resultant induction (Theorem VI, § 43) only differs by a constant numerical factor $1/4\,\pi$.

The convergence of the magnetisation is then negligible, and the magnetic action at external points arises almost exclu-

[1] This theorem constitutes the theoretical foundation of a magneto-optical method of measurement employed by the author (*Phil. Mag.* [5], vol. **29**, p. 293, 1890), and described below in § 222.

G 2

sively from the end-elements on the bounding-surface of the body. The other limiting case is that in which the values of $\mathfrak{H}_e$ and $\mathfrak{H}_i$ become indefinitely great; it is evident that this case cannot be completely realised in practice, though we may approximate to it more or less closely. In accordance with the theorem of Kirchhoff already given (III, § 57), the distribution of magnetisation is, in the limiting case, neither lamellar nor solenoidal, for it tends everywhere towards the same saturation-value, without at the same time having at all points the same constant direction, which is a necessary condition for a uniform distribution (§ 43); but the magnetisation must still have a complex-lamellar distribution, as we have already given (§ 57) a perfectly general proof of this property.

§ 60. **Ferromagnetic Body conveying a Current.**—The case in which the ferromagnetic substance itself is conveying an electric current has so far been specifically excluded from consideration; but in connection with some problems it has both a theoretical and a practical interest,[1] and accordingly we shall devote this paragraph to the question, without, however, entering upon a detailed investigation.

We have already seen that the magnetic intensity arising from the electromagnetic action of a current (and expressed, therefore, by $\mathfrak{H}_e$ in our present notation) is distributed solenoidally within the conductor, but not lamellarly, as is the case outside. It further satisfies the equation [§ 45, equation (16)]

$$(11) \quad . \quad . \quad \begin{cases} 4\pi\mathfrak{C}_x = \dfrac{\partial \mathfrak{H}_{ez}}{\partial y} - \dfrac{\partial \mathfrak{H}_{ey}}{\partial z} \\ 4\pi\mathfrak{C}_y = \dfrac{\partial \mathfrak{H}_{ex}}{\partial z} - \dfrac{\partial \mathfrak{H}_{ez}}{\partial x} \\ 4\pi\mathfrak{C}_z = \dfrac{\partial \mathfrak{H}_{ey}}{\partial x} - \dfrac{\partial \mathfrak{H}_{ex}}{\partial y} \end{cases}$$

in which $\mathfrak{C}$ denotes the electric flow (as defined in § 44) at the point considered.

To determine whether the distribution of $\mathfrak{H}_e$ is perchance complex-lamellar, we must insert the above values in the

[1] In this connection, we need only mention iron telegraph wires, and those dynamos in which the iron armature serves to convey a current, as, for example, those constructed by Fritsche. (Compare § 144.)

equation which conditions this property [§ 38, equation (6)], which then, after simplification by division, assumes the form

$$\mathfrak{C}_x \mathfrak{H}_{ex} + \mathfrak{C}_y \mathfrak{H}_{ey} + \mathfrak{C}_z \mathfrak{H}_{ez} = 0$$

This is easily seen to be equivalent to

$$\mathfrak{C} \mathfrak{H}_e \cos (\mathfrak{C}, \mathfrak{H}_e) = 0$$

which would imply that at every point the two vectors $\mathfrak{C}$ and $\mathfrak{H}_e$ must be in directions perpendicular to one another. This, however, is not, in general the case; and it is possible to imagine cases where the directions of the two vectors are inclined to one another at any angle. Thus $\mathfrak{H}_e$ has not even a complex-lamellar distribution.

Again, the distribution of $\mathfrak{H}_i$ is always lamellar (§ 47), but the vector-sum $\mathfrak{H}_t = \mathfrak{H}_e + \mathfrak{H}_i$, being thus the resultant of two vectors, one distributed solenoidally and the other lamellarly, has a distribution of no especial character; we may therefore say that its distribution is complex-solenoidal. Hence, in accordance with the principles already established, the same must be true of the magnetisation $\mathfrak{J}$, which at every point is coincident in direction with $\mathfrak{H}_t$.

Since $\mathfrak{B}_e$ is identical with $\mathfrak{H}_e$, and is therefore also distributed solenoidally, and the same was shown above (§ 52) to hold good for $\mathfrak{B}_i$, it follows that even in a ferromagnetic body through which a current is flowing the distribution of the vector $\mathfrak{B}_t$ is still solenoidal.

The solenoidal property, or, as it is also called, the continuity of the resultant induction, is thus seen to constitute a perfectly general fundamental principle,[1] which we shall consider somewhat more fully in the following paragraphs.

§ 61. **Conservation of the Flux of Induction.**—In explaining the general properties of any solenoidal distribution of a vector (§ 37) we have already shown that the whole of the region under consideration is divisible into vector-tubes, which are of such a form that the surface-integral of the vector over any cross-section whatever of the tube has a constant value. To vector-tubes possessing this property we gave the name of solenoids.

[1] Compare Janet, *Journal de Physique* [2], vol. 9, p. 500, 1890.

If we apply this result to a ferromagnetic body subject to inductive influence, the whole of infinite space must be included in the scope of our enquiry, since the distance to which the magnetic action extends is unlimited. In the special case with which we are now concerned the *tubes of induction* appear in place of the vector-tubes of the more general case, their generating lines being the *lines of induction.*

The surface-integral of the resultant magnetic induction over any cross-section of a tube of induction we shall call the *flux of induction* through the tube, denoting its value by $\mathfrak{G}_i$. This nomenclature is founded on a hydrodynamical analogy (compare Chap. VII.). Since, in the case of an incompressible fluid, the velocity (that is, the quantity of fluid flowing across a normal surface-element per unit area per unit time) is known to satisfy that form of the equation of continuity which conditions a solenoidal distribution, it follows that the surface-integral of the velocity over any area depends only on the boundary of the latter, and measures the flux of fluid, or quantity which flows per unit time, across the area in question. The tubes of magnetic induction into which the whole of space may be mapped out must be either in the form of closed solenoids, re-entering into themselves after traversing a finite distance, or else in the form of endless solenoids which proceed to infinity, spreading out wider and wider without limit. But in both cases the flux of induction corresponding to any given tube alike remains constant, so that, in the case of infinitely long tubes, as the cross-sectional area becomes indefinitely great, the induction becomes indefinitely small, the surface-integral of the induction over the cross-section thus assuming the form $\infty \times 0$, corresponding to the constant finite value of the flux of induction for the tube in question.

Thus, the fundamental principle, established in all its generality in the last section, expressing that the distribution of the resultant induction is everywhere solenoidal, and named accordingly the principle of the continuity of the resultant induction, might equally well be called *the principle of the conservation of the flux of induction.*

§ 62. **Refraction of the Lines of Induction.**—The interpretation of this principle is sufficiently obvious in any continuous

region, whether we are concerned with closed solenoids or with those which extend to infinity; but we must now further enquire what relations hold at a surface of discontinuity—that is, in the present investigation, at the surface separating a ferromagnetic body from the surrounding medium.

Owing to the discontinuity of the tangential component $\mathfrak{B}_{t_r}$, the direction of the lines of resultant induction will, in general, change abruptly at the bounding surface; we shall have what is called a *refraction of the lines of induction.*

Since $\mathfrak{H}_t$ possesses tangential continuity (§ 56), the components $\mathfrak{H}'_{t_r}$ and $\mathfrak{H}_{t_r}$, in the immediate neighbourhood of the bounding surface on its two sides respectively, will be coincident in direction; and, in accordance with what we have already proved, this must also hold good for the tangential components $\mathfrak{B}'_{t_r}$ and $\mathfrak{B}_{t_r}$ at the ferromagnetic and outer sides of the bounding-surface respectively. The tangential components of the resultant induction may be represented geometrically as the projections of this vector upon the tangent plane to the bounding-surface at the point considered (§ 34). If in addition to these we know the value of the normal component, which is at right angles to them, the vector will be completely determined; its direction must evidently lie in that plane which contains the directions of its two components. Accordingly the lines of resultant induction within and without the ferromagnetic body will lie in the same plane, which contains also the normals $\mathfrak{N}'$ and $\mathfrak{N}$; or at least this will be true when we are dealing with such short portions of the line in the immediate neighbourhood of the bounding-surface as may be considered sensibly straight. On this plane, which may by analogy be called the *plane of incidence*, $\overline{GG}$ (fig. 13) is the trace of the bounding surface, while $\mathfrak{B}'_t$ and $\mathfrak{B}_t$ are the directions of the lines of resultant induction on either side of the surface, the angles made by these lines with the normals $\mathfrak{N}'$

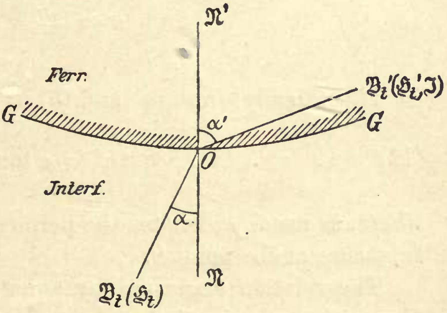

FIG. 13

and $\mathfrak{N}$ being denoted by a' and a respectively. The lines of resultant magnetic force ($\mathfrak{H}'_i$ and $\mathfrak{H}_i$) are further known to coincide with the lines of resultant magnetic induction, and, moreover, within the ferromagnetic substance the lines of magnetisation ($\mathfrak{I}$) have the same direction in common with these.

Starting from the known relations $\mathfrak{H}'_{i_\tau} = \mathfrak{H}_{i_\tau}$ and $\mathfrak{B}'_{i_\nu} = \mathfrak{B}_{i_\nu}$ (§ 58), we have at the point 0 :—

A. In the ferromagnetic substance

$$\mathfrak{B}'_{i_\tau} = \mu\, \mathfrak{H}'_{i_\tau} = \mu\, \mathfrak{H}_{i_\tau} \quad \text{and} \quad \mathfrak{B}'_{i_\nu} = \mathfrak{B}_{i_\nu}.$$

Therefore,

$$(a) \qquad \tan a' = \frac{\mathfrak{B}'_{i_\tau}}{\mathfrak{B}'_{i_\nu}} = \frac{\mu\, \mathfrak{H}_{i_\tau}}{\mathfrak{B}_{i_\nu}}$$

B. In the surrounding medium :

$$(b) \qquad \tan a = \frac{\mathfrak{B}_{i_\tau}}{\mathfrak{B}_{i_\nu}} = \frac{\mathfrak{H}_{i_\tau}}{\mathfrak{B}_{i_\nu}}$$

Thus, finally, from (a) and (b)

$$(12) \quad . \quad . \quad . \quad \tan a' = \mu \tan a$$

where, as usual, μ denotes the permeability of the ferromagnetic substance at the point 0.

The relation expressed by equation (12) may be called *the tangent-law of refraction* of the resultant induction.

In the cases which most frequently arise, μ is a large number, its value being sometimes as high as several thousand, so that the value of a is always very small, even when that of a' is considerable. In almost all cases, therefore, the lines of induction which pass from the ferromagnetic body into the surrounding medium leave the surface of separation in a direction nearly coincident with the normal.

On the other hand, as the magnetisation rises to a very high value, the value of the permeability approaches more and more nearly to unity (§ 14), and the refraction becomes continually less marked. In the limiting case, when $\mu = 1$, there will no

longer be any refraction, for then, by equation (12), we have $a' = a$.

From these rules governing the refraction at the bounding-surface of a body we may often obtain an approximate estimate of the distribution of the lines of induction in the neighbourhood of the surface.

Let us imagine two very narrow tubes of induction, whose normal sectional areas are very small and equal to $\delta S'$ and δS respectively, the directions of the tubes being coincident with those of the lines of induction ($\mathfrak{B}_i'$ or $\mathfrak{B}_i$, as the case may be); and, further, let these two tubes meet one another at the bounding-surface of the body, each intercepting there the same surface element δS_τ.

Then we shall have (fig. 13, p. 87)

$$\delta S' = \delta S_\tau \cos a' \quad \text{and} \quad \delta S = \delta S_\tau \cos a$$

Also

$$\mathfrak{B}' = \frac{\mathfrak{B}'_{i\nu}}{\cos a'} \quad \text{and} \quad \mathfrak{B}_i = \frac{\mathfrak{B}_{i\nu}}{\cos a}$$

The product $\mathfrak{B}_i' \, \delta S'$ or $\mathfrak{B}_i \, \delta S$ is evidently equal to the flux of induction through the narrow solenoids. Denoting it by $\delta \mathfrak{G}_i'$ or $\delta \mathfrak{G}_i$, we obtain for its value from the above equations, on multiplying them together in pairs,

$$\delta \mathfrak{G}_i' = \mathfrak{B}_i' \, \delta S' = \frac{\mathfrak{B}'_{i\nu}}{\cos a'} \, . \, \delta S_\tau \cos a'$$

and

$$\delta \mathfrak{G}_i = \mathfrak{B}_i \, \delta S = \frac{\mathfrak{B}_{i\nu}}{\cos a} \, . \, \delta S_\tau \cos a$$

Since $\mathfrak{B}'_{i\nu} = \mathfrak{B}_{i\nu}$, and the cosines in the last two equations cancel one another, these equations are identical, and

(13) . . . $$\delta \mathfrak{G}_i' = \delta \mathfrak{G}_i$$

From this equation we see that the principle of the conservation of the flux of induction remains valid, even at a surface where refraction takes place. It is evident that, by integration, equation (13) can be immediately extended to a solenoid whose cross-sectional area is as great as we please.

§ 63. **Representation of the Magnetic Field by means of Unit-Tubes.**—In the last chapter we adduced several considerations

concerning infinitely narrow vector-tubes (§ 37), and we further introduced the conception of the strength as the product of the value of the vector and the normal sectional area of the tube. Again, a unit-tube has been defined as one whose strength, constant throughout its entire length, is unity; and we have already explained how, in some cases, such unit-tubes may furnish a representation of the field.

In the special case with which we are now concerned, the flux of resultant magnetic induction through any tube of induction must evidently be introduced as the equivalent of the strength of the corresponding vector-tube; and the principle of the conservation of the flux of resultant induction, investigated above, may now be expressed in the following form, which depends on Theorem III, given in § 37 :—

V. *Space may be divided into infinitely narrow unit-tubes, throughout each of which the flux of induction is constant and equal to unity.*

When we are given such a system of unit-tubes (which must either form closed circuits or spread out till they lose themselves in infinity), the distribution of the induction is wholly determinate, and its character is, moreover, very clearly represented; for the direction of the unit-tube at any point gives immediately that of the induction. The 'density' of the tubes, on the other hand (that is, the number intercepted per unit area of a normal surface), is equal to the numerical value of the induction; the number of unit-tubes which intersect a given bounded surface (and which are consequently encircled by the curve bounding the surface) being numerically equal to the flux of induction through the surface in question. The flux of induction through any surface is consequently determined by the boundary of the surface alone, and is independent of the form of the surface within the boundary.

A closed surface we may imagine to be divided into two parts by a closed curve arbitrarily drawn upon the surface itself. The flux of induction through each of these two parts is the same; but through one part the flux is measured *into* the region enclosed by the surface, and through the other part it is the flux *out of* the region which we measure. As an immediate consequence we have the theorem that the flux of induction

inward through any closed surface is always equal to the flux outward, so that the algebraic sum of these two parts is zero. This property is, moreover, only a special case of that which we have already cited as defining the solenoidal distribution of a vector: that the surface-integral of the vector, taken over any closed surface whatever, must be zero (§ 37).

It is especially important to remark that the theorem already stated, as well as those which are to follow, hold good quite independently of the media through which the unit-tubes of induction extend, whether they pass wholly or partly either through magnetically indifferent or through ferromagnetic substances, so that the truth of these theorems is perfectly general.

§ 64. **Induced Electromotive Forces.**—It is one of the most fundamental properties of the flux of induction through a closed curve (or of the number of unit-tubes embraced by the curve) that its variations completely determine the electromotive forces induced in a conductor which coincides with the curve in position. In fact, the time-integral of the induced electromotive force E is numerically equal to the variation of the number of unit-tubes embraced by the circuit; or, in other words, the electro-motive force is at each instant equal to the rate of variation of the number of tubes, reckoned per unit time. This electro-motive force, divided by the total resistance of the circuit which includes the conductor in question, gives accordingly the value of the current I, flowing at any instant. If instead, we divide its time-integral by the resistance (R), we obtain the value of the 'integral current'; that is, the total quantity of electricity Q displaced round the circuit owing to the variation $\delta \mathfrak{G}_t$ in the flux of induction. Denoting the time by T, we have

$$(14) \qquad Q = \int I \, d\,T = \frac{1}{R} \int E \, d\,T = \frac{\delta \mathfrak{G}_t}{R}$$

as the equation representing the process of induction in the most general case. This result holds good in whatever manner the variation $\delta \mathfrak{G}_t$ is effected; whether by a change in the relative positions of the conductor and the system of unit-tubes, or by a change in the 'density' of the tubes; be the change of density due to a variation of the current in an

inducing primary circuit, or to any other cause whatever. The sense in which the induced current flows is determined in accordance with the general rule given by Lenz, namely that its electromagnetic action is always such as to oppose the variation $\delta\,\mathfrak{G}_t$ which induces it.

Let us apply the general equation (14) to the simple case of a conductor in the form of a plane curve enclosing an area S, the resultant induction $\mathfrak{B}_t$ through the area being uniformly distributed, and being suddenly brought into existence or made to vanish. The change $\delta\,\mathfrak{G}_t$ in the value of the flux of induction —that is, the number of unit-tubes which enter or leave the embrace of the circuit—will evidently be

$$\delta\,\mathfrak{G}_t = \pm\,\mathfrak{B}_t S$$

Substituting this in (14) we obtain

$$(15) \quad . \quad . \quad . \quad . \quad Q = \frac{\mathfrak{B}_t S}{R}$$

supposing the conductor to embrace a ferromagnetic body. If we are concerned with a magnetically indifferent medium, $\mathfrak{B}_t$ becomes identical with $\mathfrak{H}_t$, and we have

$$(16) \quad . \quad . \quad . \quad . \quad Q = \frac{\mathfrak{H}_t S}{R}$$

These two equations, (15) and (16), evidently correspond to equations (10) and (11) of Chapter I. (§§ 9, 10).

The theorems of the present section must be regarded primarily as experimental results, which we owe to the epoch-making investigations of Faraday.[1] But they may also be deduced theoretically from the electro-magnetic phenomena discovered by Oersted and Ampère by applying the principle of the conservation of energy. This was first shown by Helmholtz,[2] and was afterwards proved independently by Lord Kelvin.[3]

§ 65. **Faraday's Lines of Force.**—The method which we have

[1] Faraday, *Exp. Researches*, 1st and 2nd series; Maxwell, vol. 2, part iv. chapter iii.

[2] Helmholtz, *Uber die Erhaltung der Kraft*, Berlin, 1847; *Wiss. Abhandl.* vol. 1, p. 12.

[3] Sir W. Thomson, *B. A. Report*, 1848; *Math. and Phys. Papers*, vol. 1, p. 91. On transient electric currents, *ibid.* vol. 1, art. 62, pp. 534–553.

developed for representing the electromagnetic field is identical with that which depends on Faraday's 'lines of force,' and which has in recent years been so generally used. Within each unit-tube let us imagine a single line of induction, which is the locus of the *centroids* of all the normal cross-sections of the tnbe. The field may then be mapped out by means of these lines of induction, as well as by the corresponding tubes. We shall only need to replace the word 'unit-tube' by 'line of induction' wherever it occurs in the foregoing theorems. In the sequel we shall occasionally make use of this graphical mode of representation, especially in those cases where mathematical rigour is not so much an object as clearness in picturing the general character of the field. It is then to be remarked that the flux of induction $\mathfrak{G}_t$ through a closed curve is represented by the bundle of lines of induction which the curve embraces, the number of lines being numerically equal to the value of $\mathfrak{G}_t$. On the other hand, the induction $\mathfrak{B}_t$ at any one place is represented by the 'density' with which the lines are there packed together, the number of lines per unit cross-section measuring directly the value of $\mathfrak{B}_t$.

The expression 'line of induction' is under all circumstances preferable to 'line of force.' Even if we use the expression 'force' instead of 'intensity' (see § 4 note) it would still be incorrect in the present case to speak of magnetic force, which is identical with magnetic induction in indifferent media only, and not in ferromagnetic substances. In these latter the distribution of magnetic intensity $\mathfrak{H}_t$ is not, in general, solenoidal (§ 56); and it must be remembered that the fundamental principle of the solenoidal distribution of $\mathfrak{B}_t$, or of the conservation of the flux of induction $\mathfrak{G}_t$, constitutes the main assumption of the so-called theory of lines of foroe. Without this principle none of the conclusions of the theory would hold good.

But though the expression 'line of force,' judged from the standpoint of present-day science, is seen to be unhappily chosen, we must not on this account be led to undervalue the fruitfulness of Faraday's method of picturing the electromagnetic field, which introduced conceptions of fundamental importance, at that time entirely new, though the form in which they were stated was rather obscure and lacking in precision. It was Maxwell who

first reduced Faraday's ideas to a form in which they could be defined with geometrical precision, and developed his general views in the manner explained in the foregoing sections.

Besides applying his lines of force to represent the distribution of magnetic condition in space—for example, in connection with the induction of electromotive forces in conductors —Faraday made use of the same method as a means of determining the mechanical action between magnetised bodies at a distance from one another; and here he starts from the assumption that the forces, apparently exerted across space, are in reality transmitted by a stressed condition of the intervening medium. Faraday attributes to the lines of force a tendency to shorten and to spread out as far as possible from one another. This somewhat vague statement has also been further developed by Maxwell, who has given to it a mathematical form. In the eleventh chapter of the fourth part of his 'Treatise,' by a method which we will not here reproduce, he arrives at the following theorems from general considerations on electromagnetic energy.

In the most general case, where the resultant induction and the resultant magnetic intensity are not coincident in direction, but make an angle a with one another (the assumption being, therefore, that the body considered is, in some measure at least, magnetically rigid, possessing hysteresis, or else is aelotropic), there is a condition of stress which may be resolved into the following components:—

1. A (hydrostatic) pressure, equal in all directions, whose value, reckoned per unit area, is $\mathfrak{H}_i'^2/8\pi$.

2. A tension in the direction which bisects the angle a; the value of this, reckoned per unit area, is $\mathfrak{B}_i'\mathfrak{H}_i'\cos^2(a/2)/4\pi$.

3. A pressure, at right angles to (2), and having the value $\mathfrak{B}_i'\mathfrak{H}_i'\sin^2(a/2)/4\pi$, reckoned per unit area.

4. A torque whose value per unit volume is $\mathfrak{B}_i'\mathfrak{H}_i'\sin a/4\pi$.

If $\mathfrak{B}_i'$ and $\mathfrak{H}_i'$ are coincident in direction, as they must be in an isotropic ferromagnetic body free from hysteresis, not to speak of the magnetically indifferent surrounding medium (§ 54), considerable simplifications are immediately effected; to the more particular consideration of these we shall return later on (§ 101).

§ 66. **Statement of the Problem of Magnetisation.**—Let us sum up the results obtained in the foregoing investigation of the theory of magnetic induction. We have seen that the magnetic fields with which we have to deal have always a lamellar-solenoidal distribution, whether they arise from the action of external magnets or from conductors conveying currents (the interior of such conductors being excluded from consideration). If we introduce into such a field a ferromagnetic body of arbitrary shape, the resulting magnetisation of the body cannot be immediately determined. We can only write down the general conditions which must be satisfied by the several magnetic quantities involved, and which we have formulated both in an analytical and in a geometrical form.

We found that the magnetisation already existing in the body produces a lamellarly distributed field, which, compounded with the field already present, gives rise to a resultant field, whose distribution is also lamellar, and which possesses, therefore, a scalar potential.

We next introduced Kirchhoff's assumption, according to which both the magnetisation and the resultant induction must at each point be coincident in direction with the resultant magnetic field. Hence we deduced that these two vectors must have a complex lamellar distribution; that the resultant induction is, in addition, solenoidally distributed under all circumstances; and that the same is sensibly true of the magnetisation in all those practical cases where we may write with a sufficient degree of approximatiom

$$\mathfrak{B}_i = 4\pi\mathfrak{I}$$

From the formal expression of these conditions we obtain, quite apart from their theoretical interest, a deeper insight into the present question, for in many problems which actually arise in experimental and practical applications they furnish some suggestion as to the mode of attack. But we are still left a long way from the solution of the general problem of induction: namely, to determine completely the magnetisation of a body of arbitrary shape when the inducing magnetic field is distributed in any manner whatever.

It may, indeed, be rigorously proved, by the aid of certain

propositions in the theory of potential, that the solution of the problem proposed is unique. But this is one of those analytical niceties which are also at once evident on simple consideration, and which have long been known from actual experience to be true without constituting an advance towards solution. The number of special cases for which an exact or even an approximate solution has been obtained is, however, very small. We shall conclude by giving a brief account of these, but we must first direct our attention to several general theorems concerning similar electromagnetic systems, and useful in dealing with many questions which arise in experimental and practical work.

§ 67. **Similar Systems. Lord Kelvin's Rules.**—Let us first consider a system of electric conductors which is so increased (or diminished) in size that all its linear dimensions become n times as great as at first, the system thus remaining geometrically similar to itself. All cross-sectional and other areas will then attain to n^2 times and all volumes to n^3 times their former values, while there will evidently be no change in the number of conductors in any assigned part of the system, which remains similar to itself.

We must now compare the values of current I and current-flow $\mathfrak{C}$ in the new system with those of corresponding quantities in the old system, on the supposition that the value of the magnetic intensity $\mathfrak{H}_e$ at corresponding points is the same.

From the equations [(15), § 44]

$$\mathfrak{H}_{ex} = \frac{\partial \mathfrak{A}_{ez}}{\partial y} - \frac{\partial \mathfrak{A}_{ey}}{\partial z}, \text{ \&c. \&c.}$$

it evidently follows that the components of the auxiliary function (vector-potential) $\mathfrak{A}_e$ must be changed in the same proportion as the linear dimensions of the system; that is, they must become n times as great as at first.

It further follows from equations (14), *i.e.* from

$$\mathfrak{A}_{ex} = \iiint \frac{\mathfrak{C}_x}{r} \, dx \, dy \, dz, \text{ \&c., \&c.}$$

that the components of $\mathfrak{C}$ must have $1/n$ times their original value, since multiplication by $dx\,dy\,dz/r$ introduces the factor

$\mathfrak{n}^3/\mathfrak{n} = \mathfrak{n}^2$; and $\mathfrak{n}^2 \times 1/\mathfrak{n} = \mathfrak{n}$, which is the multiplier required in the case of the components of $\mathfrak{A}_e$. Since the components of current I are obtained by multiplying those of the current-flow $\mathfrak{C}$ by the corresponding cross-sectional areas, it follows that each current in the new system will have $\mathfrak{n}^2 \times 1/\mathfrak{n} = \mathfrak{n}$ times the value of the corresponding current in the old system. Thus we arrive at the following rule:—

VI. *In order that geometrically similar systems of conductors conveying currents may exert the same magnetic intensity at corresponding points, the values of current-flow must be inversely, and those of current itself directly, proportional to the linear dimensions.*[1]

Secondly, let us consider two geometrically similar rigid magnets, the magnetisation at corresponding points of the two being equal in value and direction. From the first term on the right hand of equation (I) § 49

$$\Upsilon_i = \iint \frac{\mathfrak{J}_\nu}{r}\, dS + \iiint \frac{\mathfrak{r}}{r}\, dx\, dy\, dz$$

it follows that the quantity dS/r, being the multiplier of the normal component of magnetisation, introduces the factor $\mathfrak{n}^2/\mathfrak{n} = \mathfrak{n}$; the second term has obviously the same dimensions as the first, and thus, finally, it follows that the magnetic potential Υ_i, in virtue of the factor $\mathfrak{n}$, is altered in the same proportion as the linear dimensions. The external magnetic intensity due to the body itself, as well as the demagnetising intensity within the body, which are derived from Υ_i by differentiation with respect to certain linear quantities, will therefore be the same in the two systems. Hence we have the following rule:—

VII. *Geometrically similar rigid magnets whose magnetisa-*

[1] If the similar systems of conductors consist of exactly the same metals, the electrical resistance R (which is proportional to length divided by cross-section) will have in the new system $1/\mathfrak{n}$ of the corresponding value in the old, so that the quantity of heat generated in a given time in accordance with Joule's law (being proportional to $I^2 R$) will have $\mathfrak{n}^2 \times 1/\mathfrak{n} = \mathfrak{n}$ times its former value. At the same time, the mass to be heated is $\mathfrak{n}^3$ times as great as before, and the exterior surface, by which heat can leave the conductor, $\mathfrak{n}^2$ times as great as before. When it is desired to produce fields of great intensity, or to obtain economically a field of given intensity, it will be worth while to consider the construction of the electromagnetic arrangement from this point of view.

tion is equal and in the same direction in corresponding points, likewise exert magnetic intensities equal and in the same direction at corresponding internal or external points.

We may now combine the rules VI and VII which have just been established, and apply them to a ferromagnetic body magnetised by electric currents—that is, to an electromagnetic system; the following rule will on consideration at once be seen to be justified:

VIII. *Geometrically similar electromagnets, in which the currents are proportional to the linear dimensions, have equal magnetisation directed in the same way at corresponding points.*

The above theorems were first given by Lord Kelvin.[1]

§ 68. **Uniform Magnetisation.**—We now pass on to the consideration of a theorem which is applicable in the case of a uniform distribution of magnetisation.

IX. *If a ferromagnetic body is of such a shape that uniform magnetisation in a determinate direction gives rise within the body to a demagnetising field, whose intensity is constant, and direction opposite to that of the magnetisation; then the component in this same direction of a uniform field due to external causes will also give rise to a uniform magnetisation of the kind already specified.*

For this external field-component, combined with the corresponding component of demagnetising intensity, gives a component of total resultant magnetic intensity which is likewise uniform in value and in the assigned direction, and so induces a component of uniform magnetisation as stated.

The theorem may be immediately extended to the case of a uniform radial or uniform peripheral distribution. The resolution of a vector into its components, and the recomposition of components into their resultant, are processes applicable to magnetic vectors, as they are to all directed quantities. If a vector is uniformly distributed, so are its components, and *vice versâ*; this follows immediately from the definition of a uniform distribution (§ 43). If, therefore, the conditions postulated in Theorem IX hold good for the directions of the three co-ordinate axes, the components of magnetisation $\mathfrak{J}_x$, $\mathfrak{J}_y$, $\mathfrak{J}_z$ will be distributed uniformly, and the same will consequently be true of

[1] Sir W. Thomson, *Repr. Pap. Electrost. and Magn.* § 564.

the resultant magnetisation $\mathfrak{J}$. Moreover, it is not necessary that this latter vector should be coincident in direction with the (uniformly distributed) resultant magnetic intensity $\mathfrak{H}_t$; this would only hold good if the components of $\mathfrak{J}$ were proportional to those of $\mathfrak{H}_t$, a condition which, generally speaking, is not fulfilled.

In general, also, $\mathfrak{H}_i$ and $\mathfrak{J}$ will not be in the same direction. In accordance with our present assumption each of the three components of the former vector depends exclusively on the component of magnetisation in its own direction, and is proportional to it, but the ratio of these two proportional components is in general different for each of the three directions of reference, so that the resultant of $\mathfrak{H}_{ix}$, $\mathfrak{H}_{iy}$, $\mathfrak{H}_{iz}$ is not coincident in direction with that of $\mathfrak{J}_x$, $\mathfrak{J}_y$, $\mathfrak{J}_z$. In analytical language, $\mathfrak{H}_{ix}$ is a function of $\mathfrak{J}_x$ alone and independent of $\mathfrak{J}_y$ and $\mathfrak{J}_z$, similar relations holding good for $\mathfrak{H}_{iy}$ and $\mathfrak{H}_{iz}$. We may therefore write:

$$\mathfrak{H}_{ix} = -N_x \mathfrak{J}_x, \quad \mathfrak{H}_{iy} = -N_y \mathfrak{J}_y, \quad \mathfrak{H}_{iz} = -N_z \mathfrak{J}_z$$

So that, in accordance with the generalised definition of § 24, N_x, N_y, N_z are the factors of demagnetisation for the directions corresponding to X, Y, Z. Hence it follows that, for the solution of the problem of the uniform magnetisation of a body (supposing such a condition to be possible), it is necessary and sufficient to know the demagnetising factors for the three principal directions in the body.

After this investigation we shall not need to adduce any separate proof, that $\mathfrak{J}$ is not in general coincident in direction with $\mathfrak{H}_e$, the external field.

§ 69. **Magnetisation of an Ellipsoid.**—It may be shown that if Γ is the gravitation potential due to a body of any shape and of uniform density D, $-\partial\Gamma/\partial x$ is the proper expression for the magnetic potential Υ_i due to the same body when magnetised uniformly, its magnetisation having the value $\mathfrak{J}_x = D$.[1]

In accordance with the theorem of the last section, in order that it be possible to induce such magnetisation (uniform throughout the body), $\mathfrak{H}_{ix}$ must necessarily be constant. But we have

$$\mathfrak{H}_{ix} = -\frac{\partial \Upsilon_i}{\partial x} = +\frac{\partial^2 \Gamma}{\partial x^2}$$

[1] Compare Maxwell, *Treatise*, 2nd edition, vol. 2, § 437.

Similarly,

$$\mathfrak{H}_{iy} = + \frac{\partial^2 \Gamma}{\partial y^2}.$$

$$\mathfrak{H}_{iz} = + \frac{\partial^2 \Gamma}{\partial z^2}.$$

But if the second differential coefficients of the gravitation-potential with respect to the co-ordinates are to be constant, this function itself must be expressible as a quadratic function of the co-ordinates. And further, from the theory of attractions it follows that this can only be the case when the attracting mass is bounded by a closed surface of the second order. The only case, again, where the mass thus bounded is of finite extent, is that in which the boundary is ellipsoidal. The problem of uniform magnetisation is thus seen to become an extremely limited one.

Let the equation of the ellipsoid considered be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

the principal axes accordingly having the same directions as the axes of reference; and let us denote by Φ the definite (elliptic) integral

$$\Phi = \int_0^\infty \frac{1}{\sqrt{(a^2 + \phi^2)(b^2 + \phi^2)(c^2 + \phi^2)}}\, d(\phi^2)$$

if now we introduce the notation

(17) $$N_x = 4\pi abc \frac{\partial \Phi}{\partial (a^2)}, \quad N_y = 4\pi abc \frac{\partial \Phi}{\partial (b^2)}$$

$$N_z = 4\pi abc \frac{\partial \Phi}{\partial (c^2)}$$

then the gravitation-potential Γ, due to matter of uniform density D filling the ellipsoid, has at points *within* the latter the value

$$\Gamma = -\frac{D}{2}(N_x x^2 + N_y y^2 + N_z z^2) + \text{const.}$$

Applying this result to the magnetic problem in the manner already explained, we obtain

$$\mathfrak{H}_{ix} = + \frac{\partial^2 \Gamma}{\partial x^2} = - N_x \mathfrak{J}_x$$

$$\mathfrak{H}_{iy} = + \frac{\partial^2 \Gamma}{\partial y^2} = - N_y \mathfrak{J}_y$$

$$\mathfrak{H}_{iz} = + \frac{\partial^2 \Gamma}{\partial z^2} = - N_z \mathfrak{J}_z$$

The quantities N_x, N_y, N_z, which are given by equation (17), are therefore the required factors of demagnetisation corresponding to the directions of the axes in the ellipsoid in question.

§ 70. **Further Special Cases.**—The formulæ (17) give the factors of demagnetisation as differential coefficients of an elliptic definite integral. These reduce to elementary functions, however, when we pass to the special case of an ellipsoid of revolution, magnetised in the direction of its axis of revolution. The formulæ for the factor of demagnetisation have already been given (§ 29) for the two distinct cases which arise, those namely of the prolate and oblate ellipsoids. It is therefore unnecessary to repeat them here. The other shapes which may be considered as special cases of the ellipsoid of revolution were the sphere, a circular cylinder of infinite length magnetised transversely, and a plate of infinite extent magnetised perpendicularly to its plane.

This last case may also be deduced from that of a thin hollow sphere, magnetised by a field which may be regarded as having approximately a uniform radial distribution. There will then also be a uniform radial distribution of magnetisation, and as the radius is made to increase indefinitely we pass from the case of the hollow sphere to that of the plane plate.[1]

The magnetisation of a circular cylinder of finite length by a field parallel to its axis was investigated by Green.[2]

Starting from several assumptions which were not, however, free from objection, he arrived at an empirical equation which

[1] du Bois, *Wied. Ann.* vol. **31**, p. 947, 1887.

[2] Green, *Essay on the Application of Mathematics to Electricity and Magnetism*, § 17, Nottingham, 1828.

is tolerably accurate for any cylinder of a length considerable in comparison with its radius. The equation gives the 'linear density of free magnetism,' along the cylinder, or, as we should express it, the normal component of magnetisation at the bounding-surface; a quantity which is zero at the middle and increases continually in numerical value as we approach either end. But this linear density is of no special interest; in experimental investigations we are chiefly concerned with the average value of the factor of demagnetisation, which we have already (Table I, p. 41) given as an empirical function of the ratio of the length to the diameter.

In addition to the problems in magnetisation already mentioned, wherein we assume the application of a uniform external field, several cases have been investigated in which the magnetising field may be distributed in any manner.

Poisson treated in this way the case of a hollow sphere by means of spherical harmonics,[1] which had already been experimentally examined by Barlow.[2] The most interesting result of these investigations is that a hollow sphere of ferromagnetic material (provided its thickness is not too small) when magnetised by any field whatever, exerts the same effect at all external points as if it were solid throughout. In the inner hollow space, on the other hand, the original external field suddenly generated will be greatly reduced by the action of the ferromagnetic shell.[3]

Again F. Neumann has investigated the magnetisation of an ellipsoid of revolution under the influence of an arbitrarily distributed external magnetic field, especially for the particular case where the ellipsoid degenerates into a sphere.[4] The investigation has further been extended by Kirchhoff to the case of an infinitely long cylinder.[5] For the details of these

[1] See Maxwell, *Treatise*, 2nd edition, vol. **2**, §§ 431–434.

[2] Barlow, *Essay on Magnetic Attractions*, London, 1820; *Gilb. Ann.* vol. **73**, p. 1, 1828.

[3] This general property of enclosures surrounded by thick shells of ferromagnetic material is often applied to diminish the effects of external magnetic influences and disturbances; for example, in marine galvanometers and other apparatus.

[4] F. Neumann, *Crelle's Journal*, vol. **37**, 1848; *Vorlesungen über Magnetismus*, § 43, p. 112, Leipzig, 1885.

[5] Kirchhoff, *Gesammelte Abhandlungen*, p. 193.

mathematical researches, as well as of many others to which reference cannot here be made, we must refer to treatises which give a complete historical survey of the present subject.[1]

§ 71. **Solution by successive Superposition.**—The inherent difficulty of the problem of magnetisation arises from the fact that the demagnetising intensity must be taken into account, and this in turn depends on the distribution of magnetisation which we have to determine. Hence some investigators have applied to this problem a method of successive approximation, which in principle is analogous to that which Murphy employed in electrostatic problems for calculating the distribution of electricity.[2] We will conclude this chapter with a brief discussion of the method.

Let the distribution of the external field $\mathfrak{H}_e$, or its magnetic potential Υ_e, be given. Neglecting for a moment the demagnetising intensity, we calculate the magnetisation which would be induced in the body considered under the influence of the potential Υ_e alone. Let this be $\mathfrak{J}'$; it may be called the magnetisation *of the first order*. This will give rise to a potential of the first order Υ_i', which in its turn will induce a magnetisation *of the second order* $\mathfrak{J}''$. This again will produce a potential of the second order Υ_i'', which in turn will cause a magnetisation *of the third order*, and so on.

The successive (vectorial) superposition of these magnetisations of different orders gives with a continually increasing degree of approximation the actual distribution of magnetisation $\mathfrak{J}$, so that

$$(18) \qquad \mathfrak{J} = \mathfrak{J}' + \mathfrak{J}'' + \mathfrak{J}''' + \ldots$$

If this method is to be suited for solving problems, we must seek to express the magnetisations of successive orders by means of the most rapidly converging series possible. This question was first considered by Beer, to whom the method is due, and subsequently by C. Neumann, L. Weber and Riecke.[3]

[1] Especially Wiedemann, *Lehre von der Elektricität*, vol. **3**, pp. 354–390; 'Nachtrag,' p. 1320.

[2] Mascart and Joubert, *Electricity and Magnetism*, vol. **1**, § 86; Wiedemann, *Lehre von der Elektricität*, vol. **1**, § 85; vol. **3**, § 387.

[3] Beer, *Einleitung in die Lehre von der Elektricität u. d. Magnetismus*, pp. 155–165; C. Neumann, *Das Logarithm. Potential*, p. 248; L. Weber, *Zur*

Quite recently Wassmuth has shown how all these expansions can be deduced from a common form, whose physical interpretation is easy.[1]

The greater number of the above researches are chiefly of mathematical interest; at least their physical results bear no sort of proportion to the mathematical skill and ingenuity that have been lavished on them. But amongst them are several results which furnish us with valuable guidance for the prosecution of experimental enquiries. For the greater number of the important modern applications of electromagnetism they are, on the other hand, entirely unfruitful. We shall now turn our attention to problems whose solution appears to promise more in this respect.

Theorie der magnetischen Induktion, Kiel, 1877; Riecke, *Wied. Ann.* vol. **13**, p. 465, 1881.

[1] Wassmuth, 'Lösung des Magnetisirungsproblems durch Reihen,' *Wiener Berichte*, vol. **102**, part 2, p. 65, 1893; *Wied. Ann.* vol. **51**, p. 367, 1894.

CHAPTER V

MAGNETISATION OF CLOSED AND OF RADIALLY DIVIDED TOROIDS

A. *Theory*

§ 72. **Peripheral Magnetisation of a Solid of Revolution.**—In the first two chapters we have already considered, in an elementary manner and from different points of view, the magnetic properties of closed rings, as well as of those which are divided by a radial slit (§§ 9, 10, 16), and we have seen how these may in a measure be considered as typical forms. The problem of their magnetisation must now be more closely examined in the light of the results obtained in the last two chapters, since this problem furnishes a basis for the further elaboration of the theory of the magnetic circuits.

Kirchhoff[1] was the first to investigate mathematically the magnetisation of a ring, or, to speak more definitely, of a solid of revolution which is not intersected by its axis of symmetry. It is assumed that each single winding of the magnetising coil lies in a 'meridian plane' which passes through the axis; all the windings, taken collectively, constitute a hollow ring, which encloses the ferromagnetic solid of revolution, and is disposed symmetrically with respect to the same axis. Let n denote the number of windings, and I the current flowing in the circuit. Any arbitrary closed curve which runs completely round within the hollow annular space is then evidently embraced n times by the current-conductor; so that when we pass once round such a path of integration, the line-integral of the magnetic intensity $\mathfrak{H}_e$, produced by the current within the region in question, increases by the value $4\pi n I$ (§ 45). In particular, let us choose as paths of integration, circles whose centres lie on the axis $\overline{ZZ}$ of the

[1] Kirchhoff, *Gesammelte Abhandlungen*, p. 223.

body of revolution (fig. 14) and whose radius may be denoted by r. It will then be evident, from considerations of symmetry, that $\mathfrak{H}_e$ must be at each point peripheral in direction—that is, tangential to the circle of integration and at right angles to the meridian plane (plane of the paper in fig. 14); while along the circumference of any one and the same circle of integration the value of $\mathfrak{H}_e$ must be constant. The line-integral in question may therefore be expressed as the product of the value of the vector into the circumference $2\pi r$ of the circle of integration.

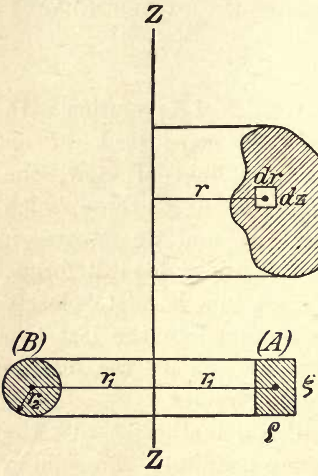

FIG. 14

Thus we have

$$2\pi r \mathfrak{H}_e = 4\pi n I$$

or

$$(1) \quad . \quad . \quad \mathfrak{H}_e = \frac{2nI}{r}$$

The magnetic intensity is therefore inversely proportional to the distance of the point considered from the axis, so that its value diminishes as the point is taken further from the axis.

§ 73. **Kirchhoff's Theory.**—Whatever be the form of the meridional section of the solid of revolution, we may conceive it to be divided up into rectangular elements $dr\,dz$ (fig. 14), each of these, on rotation about the axis $\overline{ZZ}$ generating an elementary ring, for which $\mathfrak{H}_e$ has then a sensibly 'uniform peripheral' distribution (§ 43). We next assume that this would give rise to a uniform peripheral distribution of magnetisation, so that there will be no demagnetising influence, magnetically effective end-elements being absent.[1] The resultant magnetic intensity is therefore identical with that ($\mathfrak{H}_e$) which is directly due to the current in the coil, and must consequently, like the latter vector, have a uniform peripheral

[1] Since a uniform peripheral distribution, as explained in the place already quoted, is also a solenoidal one, the convergence of the magnetisation will be everywhere zero, and there will consequently be no internal centres of action which can produce any effect at distant points (§§ 37, 50).

distribution. Thus it follows from the assumption that we have made, in accordance with a theorem already given (§ 68), that there is a uniform peripheral distribution of magnetisation within the thin elementary ring. Just as there is an absence of demagnetising effect within the elementary ring, so also this latter remains without magnetic influence at external points, or upon neighbouring elementary rings, which are thus seen to be wholly independent of one another as regards their magnetic properties. Throughout the entire ring, therefore, the magnetisation will have a constant value for each given value of the radius, but will decrease with $\mathfrak{H}_e$ as the radius increases; it is given by an equation which follows directly from (1):—

$$(2) \quad . \quad . \quad \mathfrak{I} = \kappa \mathfrak{H}_t = \kappa \mathfrak{H}_e = \frac{2 n \kappa I}{r}$$

where κ denotes the susceptibility, which is variable and a function of $\mathfrak{H}_e$.

Similarly, the resultant magnetic induction is given by the equation

$$(3) \quad . \quad . \quad . \quad \mathfrak{B}_t = \frac{2 n \mu I}{r}$$

where μ denotes the (variable) permeability. Finally, we have for the total flux of induction through the meridional cross-section of the ring (that is, the quantity whose variations [§ 64] determine the electromotive forces induced in a secondary wire wound closely over the ring)

$$(4) \quad . \quad . \quad \mathfrak{G}_t = 2 n I \iint \frac{\mu \, dr \, dz}{r}$$

the double integration being extended over the entire cross-section of the ring. Since μ is not known as a function of r, this integration cannot, strictly speaking, be effected. In general, the variation of μ as we pass from the inner to the outer portions of the ring is much slower than that of the reciprocal of the radius, and it follows from the rules for the evaluation of definite integrals that in each case we may replace μ by a certain mean value $\bar{\mu}$, outside the sign of integration. When this is done there are a number of cases in which the integration can be performed.

§ 74. **Rings of Rectangular and of Circular Section.**—The results are easily calculated for rings of rectangular or circular section, and these we now proceed to give.

(A) *Rectangular section* [fig. 14 (*A*), p. 106].—Let ζ be the height of the cross-section, ρ its breadth in the radial direction, r_1 the mean radius, reckoned from the axis of symmetry $\overline{ZZ}$ to the centre of the rectangle. Then we have

$$(5) \quad . \quad . \quad \mathfrak{G}_i = 2\, n\, \bar{\mu}\, I\, \zeta \log \frac{2\, r_1 + \rho}{2\, r_1 - \rho}$$

If ρ is very small compared with r_1, then to a sufficient degree of approximation

$$\frac{1 + \dfrac{\rho}{2\, r_1}}{1 - \dfrac{\rho}{2\, r_1}} = 1 + \frac{\rho}{r_1}$$

We may now in the usual manner expand $\log \left(1 + \dfrac{\rho}{r_1}\right)$ in the form of a series, of which only the first term need be retained. Thus,

$$(6) \quad . \quad . \quad \mathfrak{G}_i = \frac{2\, n\, \bar{\mu}\, I\, \zeta\, \rho}{r_1} = \frac{2\, n\, \bar{\mu}\, I\, S}{r_1}$$

where S denotes the area of the cross-section. A ring of rectangular section for which $\zeta > \rho$ may be called a *hoop* (Reifring), while a *flat ring* (Flachring) is one for which $\rho > \zeta$.

(B) *Circular section* [fig. 14 (*B*), p. 106].—Our solid oi revolution now comes under the definition of a toroid (§ 9). Let r_2 be the radius of the cross-section, r_1, as before, the radius of the circular axis—that is, of the circle which is the locus of the centres of all the (circular) cross-sections. Then

$$(7) \quad . \quad . \quad \mathfrak{G}_i = 2\, n\, \bar{\mu}\, I\, 2\, \pi\, (r_1 - \sqrt{r_1^2 - r_2^2})$$

Here again, if r_2 is small in comparison with r_1, the expression approximates to the following simple form:—

$$(6a) \quad . \quad . \quad \mathfrak{G}_i = \frac{2\, n\, \bar{\mu}\, I\, \pi\, r_2^2}{r_1} = \frac{2\, n\, \bar{\mu}\, I\, S}{r_1}$$

which is identical with that given by equation (6) above.

Kirchhoff, in the paper already quoted, expressed the opinion that this special case of magnetisation, first theoretically investigated by himself, might form the basis for a convenient method of practical measurement. We shall return to the more particular consideration of this method in § 83, where it will be illustrated by an example. Kirchhoff's proposal was first realised by Stoletow in 1872; and soon afterwards Rowland conducted a series of fruitful researches by means of the method, which has since been made the basis for a large number of investigations connected with this subject.[1]

§ 75. **Fundamental Equation for a Radially Divided Toroid.**—We shall here confine our attention to the case of a ring of circular section—that is, of a toroid—and we shall assume that the dimensions of the cross-section are small in comparison with the diameter of the toroid, so that r_2/r_1 is a small quantity, as we have already supposed it to be in equation (6a). Let us now make a radial cut through the toroid, the resulting slit having throughout a constant width, which we denote by d (fig. 15). The slit is supposed to have no influence on the regularity of the winding, which we assume to be entirely uniform as before. If n is the number of windings, the intensity of the magnetic field exerted by the current at all points of the circular axis (dotted in fig. 15) will be, in accordance with equation (1),

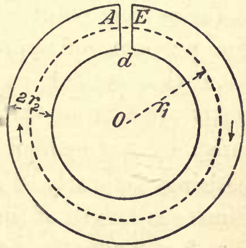

FIG. 15

$$\mathfrak{H}_e = \frac{2\,n\,I}{r_1}$$

[1] Stoletow, *Pogg. Ann.* vol. **146**, p. 442, 1872; Rowland, *Phil. Mag.* [4], vol. **46**, p. 140, 1878. Up to the present time these investigations have given us no reason to doubt the correctness of Kirchhoff's theory. Researches of G. vom Hofe (Diss. Greifswald, 1889; *Wied. Ann.* vol. **37**, p. 482, 1889) on three rings of rectangular section, for which the ratio $\rho : \zeta$ was different, appeared to show some divergence from the theory, but these can be fully explained by the differences which are known often to exist between materials which are supposed to be identical; such differences arising from the manner in which the substances have been heated and annealed, as well as from a variety of causes less easily recognised. Compare also Mues, *Über den Magnetismus von Eisenringen*, &c., p. 2 (Diss. Greifswald, 1893).

and will not differ greatly from this value at any point of the cross-section.

Let us now choose the centroid as the path of integration, and apply Theorem I, already enunciated [§ 55, equation (6)] which states that the increase of the magnetic potential due to magnetisation

$$(8) \quad . \quad . \quad . \quad {}^{A}_{E}\Upsilon_i = \int_E^A \mathfrak{H}'_{iL}\, dL$$

In the present case the points E and A are those in which the circular axis intersects the two faces of the slit. From considerations of symmetry we see that the circular axis must be a line of magnetisation—that is, at each point of the circular axis the direction of the tangent will coincide with that of the vector $\mathfrak{I}$. Moreover, the value of $\mathfrak{I}$ will not vary more perceptibly from point to point along the circular axis, any more than it varies from point to point over the cross-section of the toroid. For purposes of approximate calculation therefore—and more than this cannot be effected in the present case—we may introduce a mean value of the magnetisation, which we distinguish by the symbol $\overline{\mathfrak{I}}$. The same holds good for the demagnetising intensity $\mathfrak{H}'_i$, whose mean value is denoted by $\overline{\mathfrak{H}'_i}$. Since the length of the path of integration from E to A through the ferromagnetic body is $(2\pi r_1 - d)$, we have in accordance with the ordinary rules for the evaluation of definite integrals

$$(9) \quad . \quad . \quad \int_E^A \mathfrak{H}'_{iL}\, dL = \overline{\mathfrak{H}'_i}\,(2\pi r_1 - d)$$

We may consider this equation as the definition of the mean value $\overline{\mathfrak{H}'_i}$. If we further introduce a mean factor of demagnetisation, defined by the equation

$$(10) \quad . \quad . \quad . \quad \overline{\mathfrak{H}'_i} = -\overline{N}\,\overline{\mathfrak{I}}$$

we obtain finally from the three equations (8), (9) and (10) above

$$(\mathrm{I}) \quad . \quad {}^{E}_{A}\Upsilon_i = -\,{}^{A}_{E}\Upsilon_i = \overline{N}\,\overline{\mathfrak{I}}\,(2\pi r_1 - d)$$

This elementary formula may be in a certain sense regarded

as the fundamental equation for a radially divided toroid.[1] The left-hand member denotes the magnetic difference of potential due to magnetisation measured within the ferromagnetic body between the two faces of the slit; it is approximately equal to the mean value $\overline{\mathfrak{H}}_i$ which the magnetic intensity due to magnetisation has within the slit, multiplied by its width.

§ 76. **First Approximation; Limiting Case.**—Another expression must now be found for the left-hand member of the fundamental equation (I). To this end, in our first approximation we make the assumption that the magnetisation $\mathfrak{J}$ is constant over the entire cross-section of the toroid, and at right angles to the plane of the section, so that it has, in accordance with the definition of § 43, a uniform peripheral distribution.[2]

According to the 'law of saturation' III (§ 57) the actual state of things must approximate more or less closely to this limiting case of our assumption, when we suppose the intensity $\mathfrak{H}_e$ of the inducing magnetic field to increase without limit, so that finally its value becomes very great in comparison with that of the demagnetising intensity $\mathfrak{H}'_i$. In accordance with this assumption there will be no magnetically effective end-elements on the convex bounding-surface of the toroid, such elements being confined to the two plane surfaces which bound the slit on either side. These will produce an external field determined by their magnetic strength. But this latter quantity has per unit area of the surface in question the value $\mathfrak{J}_\nu$ (§ 49), and since, in the present case the magnetisation is at right angles to the bounding-surfaces of the cleft, $\mathfrak{J}_\nu = \mathfrak{J}$.

Let us now consider an element of one of the faces of the slit, taken in the form of a plane circular ring of infinitesimal breadth dy and of mean radius y (fig. 16); its area is $2\pi y\,dy$, and its magnetic strength is therefore $2\pi \mathfrak{J} y\,dy$. Thus at a point P which is on the normal, drawn from the centre of the bounding-face of the slit, at a distance x from that face, our elementary plane ring exerts an infinitesimal magnetic intensity

[1] du Bois, *Wied. Ann.* vol. **46**, p. 494, equation (7), 1892.

[2] It is unnecessary in this case to introduce a mean value $\overline{\mathfrak{J}}$, since the value of $\mathfrak{J}$ is everywhere the same.

$d\,\mathfrak{H}_i$, which in accordance with Coulomb's law is given by the equation

$$(11) \qquad . \qquad . \qquad d\,\mathfrak{H}_i = \frac{2\,\pi\,\mathfrak{J} y\,dy}{z^2}\cos a.$$

Here $z^2 = x^2 + y^2$ is the distance of each separate point (such as Q) of the elementary plane ring from P. The angle

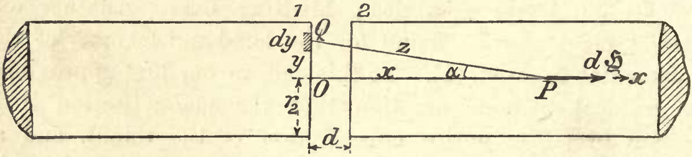

FIG. 16

$Q\,P\,O$ is denoted by a, so that $\cos a = x/z$; and consequently the above equation can be written in the following form:

$$(12) \qquad . \qquad . \qquad . \; d\,\mathfrak{H}_i = \frac{2\,\pi\,\mathfrak{J} y\,x\,dy}{z^3}$$

From the relation

$$z^2 = x^2 + y^2$$

it follows that, for a given position of the point P, that is, for a constant value of x,

$$z\,dz = y\,dy$$

And if we substitute this in (12), we obtain

$$d\,\mathfrak{H}_i = \frac{2\,\pi\,\mathfrak{J}\,x\,dz}{z^2}$$

This expression has to be integrated over the entire surface which forms one face of the slit, in order to find the resultant magnetic intensity $\mathfrak{H}_i$ at P, arising from the action of that face; we shall have, then,

$$\mathfrak{H}_i = 2\,\pi\,\mathfrak{J}\,x \int_x^{\sqrt{x^2 + r_2^2}} \frac{1}{z^2}\,dz$$

The two limits of this definite integral correspond to the centre and the circumference of the surface in question; performing the integration we obtain

$$\mathfrak{H}_i = 2\,\pi\,\mathfrak{J}\,x \Big[-\frac{1}{z}\Big]_r^{\sqrt{x^2 + r_2^2}}$$

or

$$(13) \quad . \quad . \quad \mathfrak{H}_i = 2\pi\mathfrak{J}\left\{1 - \frac{x}{\sqrt{x^2 + r_2^2}}\right\}$$

We are now in a position to calculate the change of magnetic potential in passing from the centre of the surface 1 to the point P: denoting this quantity by Υ_{i1}, we have

$$(14) \, . \quad \Upsilon_{i1} = \int_0^x \mathfrak{H}_i \, dx = 2\pi\mathfrak{J}\left\{x - \int_0^x \frac{x\,dx}{\sqrt{x^2 + r_2^2}}\right\}$$

If we put $x^2 + r_2^2 = u^2$, so that r_2, being a constant, $u\,du = x\,dx$, and if we then take u as a new variable under the last sign of integration, the limits of the definite integral become r_2 and $\sqrt{x^2 + r_2^2}$, and we obtain

$$\int_{r_2}^{\sqrt{x^2 + r_2^2}} du = \sqrt{x^2 + r_2^2} - r_2$$

the substitution of this in (14) gives

$$(15) \quad . \quad \Upsilon_{i1} = 2\pi\mathfrak{J}\,(x + r_2 - \sqrt{x^2 + r_2^2})$$

Let us now return to the consideration of the slit, that is of the region bounded by the pair of surfaces 1 and 2, whose distance apart is d (fig. 16, p. 112); in (15) we must put $x = d$, and since ${}_A^E\Upsilon_i = \Upsilon_{i1} + \Upsilon_{i2}$ and the magnetic fields due to the two surfaces are directed in the same sense, we have

$$(16) \quad . \quad {}_A^E\Upsilon_i = 4\pi\mathfrak{J}\,(d + r_2 - \sqrt{d^2 + r_2^2})$$

Thus we have found a second expression for the left-hand member of the fundamental equation (I);[1] inserting it in its proper place we obtain

$$4\pi\mathfrak{J}\,(d + r_2 - \sqrt{d^2 + r_2^2}) = \bar{N}_\infty \mathfrak{J}\,(2\pi r_1 - d)$$

[1] In the course of the proof of this expression given by the author (*Wied. Ann.* vol. 46, p. 494, equation (8), 1892), an incorrect term

$$2\int_0^d \frac{x\,dx}{\sqrt{r_2^2 + x^2}}$$

crept in through an error in transcription; this has, however, no influence on the result there obtained.

The suffix $_\infty$ indicates that the value $\overline{N}_\infty$, strictly speaking, is only attained when the intensity of the magnetising field becomes infinitely great, for it is then alone that our initial assumption holds good. If we now divide the last equation by the factors of $\overline{N}_\infty$, we obtain for this limiting value of the factor of demagnetisation

$$\text{(II)} \quad . \quad . \quad \overline{N}_\infty = \frac{2\left(d + r_2 - \sqrt{d^2 + r_2^{\,2}}\right)}{r_1 - \dfrac{d}{2\pi}}$$

In accordance with the assumption made at the beginning of this section, the values of $\overline{N}$ as measured experimentally must approach the value of $\overline{N}_\infty$ given by the expression (II) as the intensity of the magnetising field is made to increase. In practice, however, we cannot reach the limiting case of a uniform peripheral distribution of magnetisation, for the intensity of field with which we can actually work is limited by the overheating of the magnetising coil, and amounts to only a few hundred C.G.S. units. To what extent the approximation is applicable in the most favourable case is a question which must be decided by experiment (compare § 89).

§ 77. **Divergence of the Lines of Induction.**—Ordinarily speaking, nearly all the cases with which we have to deal will be those in which we can make use of the assumption of § 59, namely, that to a sufficient degree of approximation $\mathfrak{J} = \mathfrak{B}_i / 4\pi$; hence the former quantity may be regarded as having a solenoidal distribution, for under all circumstances the latter possesses this property. Now the distribution of $\mathfrak{B}_i$ can be everywhere represented by means of the lines of induction, and the same lines may therefore be made to furnish a complete representation of the distribution $\mathfrak{J}$, for the constant factor 4π constitutes the only difference between these two quantities. The simplification thus resulting from our assumption makes it far easier to obtain a general notion of the relations which obtain in our problem. We shall also find our work simplified by remarking that, when the intensity of the magnetising field is small, the permeability may always be regarded as a large number.

Let us consider from this point of view the distribution of $\mathfrak{I}$ and $\mathfrak{B}_t$ in a divided toroid. The demagnetising effect due to the presence of magnetically effective end-elements at the bounding-faces of the slit will be most perceptible in the neighbourhood of the slit itself. In the foregoing we have generally taken account only of the mean effect over the whole surface of the toroid. The magnetisation, and therefore also the resultant induction, will for this reason have somewhat smaller values in

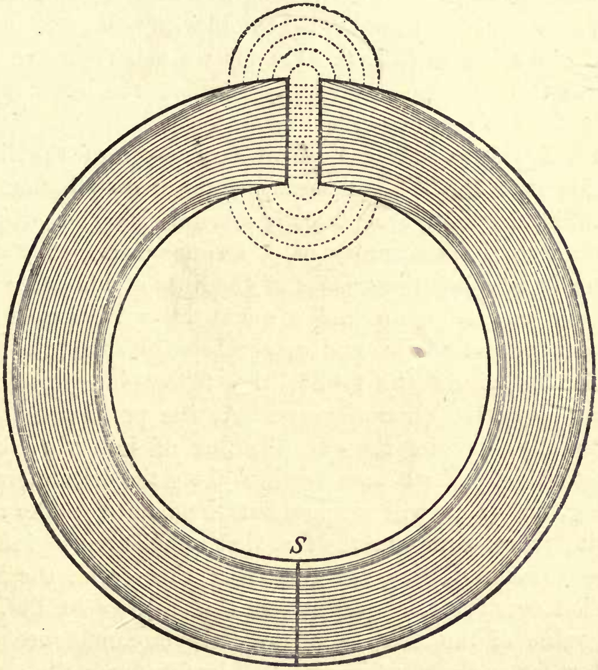

FIG. 17

the neighbourhood of the slit; and this we may picture by the fact that the tubes of resultant induction, which along the greater part of the circumference of the toroid are almost exactly in the form of co-axial circles, spread out as they approach the slit, so that there is there a divergence among the lines of induction. The lines nearest to the convex surface of the ring meet it at an acute angle, and then proceed into the surrounding medium in a direction nearly coincident

with the normal. This follows from our investigation concerning the tangent-law of refraction of lines of induction in the special case (§ 62), where the permeability has a relatively high value, a condition which we have here assumed to be fulfilled. Fig. 17 gives a diagrammatic representation of the lines of induction within a ferromagnetic radially-divided toroid, as well as in the surrounding medium; these latter portions of the lines, which are dotted in the figure, are identical with the lines of magnetic intensity. It is interesting to compare the diagram of the external field due to a divided toroid with that corresponding to a straight bar (fig. 7, p. 31); for we may conceive of the toroid as formed from the bar by bending the latter together into the form of a nearly closed hoop.

§ 78. **Leakage-Coefficient.**—This divergence of the lines of induction near the slit and their manner of distribution in the surrounding medium, are perfectly general characteristics of all places where the continuity of a ferromagnetic substance is broken; it is called the *leakage* of the lines of induction. We must now endeavour to find a quantitative measure for this phenomenon, and to this end we shall consider the total flux of induction $\mathfrak{G}'_i$ within the toroid, and follow its variations as we pass along the circumference. At the position S (fig. 17) diametrically opposite the slit, the flux of induction has its greatest value. If we pass further along the circumference of the ring, the value will change but little until we are near to the slit; there, owing to the fact that some tubes of induction emerge through the curved surface of the toroid, the flux of induction over the cross-section will evidently be smaller. The mean value of the flux, taken over the circumference of the toroid, so far as the latter is occupied by ferromagnetic material, we shall denote by the symbol $\overline{\mathfrak{G}'_i}$. Within the slit itself the flux of induction $\mathfrak{G}_i$ has evidently its smallest value, for it is there that the highest proportion of the tubes of induction pass outside the surface over which the flux is reckoned; that is, they pass outside the cylindrical space whose ends are the faces of the slit. Finally, the ratio of the mean flux of induction to the flux within the slit is to be taken as the measure of the leakage; that is, we write

$$(17) \quad . \quad . \quad . \qquad \nu = \frac{\overline{\mathfrak{G}'_i}}{\mathfrak{G}_i} \qquad [\nu \geq 1]$$

This number ν must necessarily be at least as great as unity, and will, in general, be greater. It is called the *leakage-coefficient.*

§ 79. **Magnetic End-elements on the Bounding-Surface.**—We have seen that the tubes of induction (and consequently the tubes of magnetisation which we have here supposed to be identical with them) intersect not only the faces of the slit, but also, to some extent, the curved surface of the toroid (fig. 17); and hence it follows that magnetic end-elements are not entirely confined to the faces of the slit, but extend also over the curved surface, being always strongest, however, in the immediate neighbourhood of the former. Remembering our remark that the distribution of the magnetisation is sensibly solenoidal, it also follows that the magnetic end-elements will be confined to the surface (faces of the slit and adjacent portions of the curved surface), and that within the ferro-magnetic substance, where the convergence of the magnetisation is everywhere zero, there will be no centres of magnetic action (§ 50).

So far, we have always assumed that r_1 is large compared with r_2; in other words, that the curvature of the circular axis of the toroid is small compared with that of the circumference of its cross section. Let us now neglect the former curvature entirely—that is, let us suppose it to be indefinitely small, so that its reciprocal, the radius r_1, is indefinitely large. Our radially divided toroid is now transformed into two circular cylinders of radius r_2, each limited in one direction and unlimited in the other, the ends of the two being at the distance d apart, as represented in fig. 16, p. 112. These cylinders we may suppose to be magnetised by a corresponding infinitely long coil.

Our first approximation was calculated on the assumption that the magnetic end-elements were entirely confined to the plane circular faces bounding the slit. For a second approximation, which we now proceed to consider, the end-elements of the curved surface must also be taken into account. The two modes of treatment have this much in common, that in each case, in seeking for the centres of action, we only take account of the slit and its immediate neighbourhood. So far

as this question is concerned, the remaining portions of the body may be entirely disregarded, a fact which has already been expressed by saying that we may regard the toroid as transformed into two circular cylinders, each unlimited in one direction.

§ 80. **Second Approximation.**—In equation (16), p. 113, we found a *first* approximation, which we repeat here in a slightly modified form (the factor d being placed outside the bracket):

$$(18) \quad . \quad {}_A^E\Upsilon_i = 4\pi\mathfrak{J}\,d\left(1 + \frac{r_2}{d} - \sqrt{1 + \left(\frac{r_2}{d}\right)^2}\right)$$

The question now arises, by what function of d and r_2 must we replace the bracketed expression in this equation in order to obtain a *second approximation*. This problem has a unique solution, which cannot, however, be immediately calculated. Since the shape of the slit is determined by the ratio r_2/d, the unknown function must also depend upon this quantity, which, together with the value of $\bar{\mathfrak{J}}$, must determine the function completely. This latter we shall accordingly denote by the symbol $\mathfrak{n}\,(r_2/d, \bar{\mathfrak{J}})$. As we shall see later on, the form of the function, within a certain range, is found empirically to be hyperbolic. On substituting this function in equation (16), we obtain for the change in the potential due to magnetisation as we pass from one face of the slit to the other

$$(19) \quad . \quad . \quad {}_A^E\Upsilon_i = 4\pi\bar{\mathfrak{J}}\,d\,\mathfrak{n}\left(\frac{r_2}{d}, \bar{\mathfrak{J}}\right)$$

This value must now be substituted for the left-hand member of our fundamental equation (I) § 75. Thus we obtain, by a simple transformation,

$$(\text{III}) \quad . \quad . \quad . \quad \bar{N} = \frac{2\,d\,\mathfrak{n}\left(\frac{r_2}{d}, \bar{\mathfrak{J}}\right)}{r_1 - \frac{d}{2\pi}}$$

This expression for the mean factor of demagnetisation of a radially divided toroid holds good, then, to a second degree of approximation. The numerator depends exclusively on the

form of the slit as such, while the denominator is proportional to the remainder of the circumference. Generally speaking, the second term of the denominator will be negligible in comparison with the first.

§ 81. **Toroid with several Radial Slits.** — The general Theorem I [§ 55, equation (6)], which is the starting-point for our deduction of the fundamental equation for a toroid radially divided by a single slit, has already been extended, in the place just quoted, to cases of a more complex nature. We obtained the generalised equation (6*a*), p. 108 (or Theorem I, A) for the case where the path of integration has more than one portion lying within the magnetically indifferent medium. We proceed to apply this theorem to a toroid which is radially divided by a number of slits 1, 2, ... *n*. The quantities corresponding to each slit will be distinguished by symbols which have the number of the slit as suffix. The separate ferromagnetic portions of the toroid we imagine to be connected rigidly together by any arrangement of magnetically indifferent material. We proceed to deduce the modified fundamental equation applicable to the present case, following the same method of treatment as in § 75.

As before, we introduce a symbol $\overline{\mathfrak{J}}$ for the mean value of the magnetisation along the entire circumference, and a corresponding symbol $\overline{\mathfrak{H}_i}$ for the mean demagnetising intensity, which latter we define as the sum of the line-integrals of demagnetising intensity along those arcs of the circular axis which lie within the ferromagnetic substance, divided by the sum of these arcs. The sum of the arcs in question is $(2\pi r_1 - \Sigma d)$, and our definition may be expressed by the following equation, which is analogous to equation (9) of § 75:—

$$(20) \quad . \quad . \quad . \quad \overline{\mathfrak{H}_i} = \frac{\Sigma \int \mathfrak{H}_{iL}\, dL}{2\pi r_1 - \Sigma d}$$

The sign of summation Σ here refers to all the slits 1 . . . . *n*, and the line-integrals are to be taken along those arcs of the circular axis which lie within the ferromagnetic substance. To define the mean factor of demagnetisation $\overline{N}$,

we must further introduce an equation analogous to (10), namely,

$$(21) \quad . \quad . \quad \overline{N} = \frac{-\overline{\mathfrak{H}'_i}}{\overline{\mathfrak{J}}} = \frac{-\Sigma \int \mathfrak{H}'_{iL}\, dL}{\overline{\mathfrak{J}}\,(2\,\pi\, r_1 - \Sigma\, d)}$$

For shortness, let us denote the changes in the magnetic-potential-due-to-magnetisation, on crossing the several slits, by $\Delta_1 \Upsilon_i$, $\Delta_2 \Upsilon_i \ldots \Delta_n \Upsilon_i$, and their sum by $\Sigma \Delta \Upsilon_i$. Then the equation (6*a*) of p. 78 may be written in the following form :—

$$(22) \quad . \quad . \quad . \quad -\Sigma\, \Delta\, \Upsilon_i = \Sigma \int \mathfrak{H}'_{iL}\, dL$$

Substituting this in (21), we obtain

$$(\text{IV}) \quad . \quad . \quad \Sigma\, \Delta\, \Upsilon_i = \overline{N}\, \overline{\mathfrak{J}}\, (2\,\pi\, r_1 - \Sigma\, d)$$

as the fundamental equation for a multiply-divided toroid. It is evidently quite analogous to that already given [(I) § 75] for a singly-divided toroid.

We may now pass on at once, as in the last section, to the second approximation, introducing for each slit a function analogous to $\mathfrak{n}$, so that

$$(23) \quad . \quad . \quad . \quad \begin{cases} \mathfrak{n}_1 = \text{funct.}\left(\dfrac{r_2}{d_1}, \overline{\mathfrak{J}}\right) \\ \mathfrak{n}_2 = \text{funct.}\left(\dfrac{r_2}{d_2}, \overline{\mathfrak{J}}\right) \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \mathfrak{n}_n = \text{funct.}\left(\dfrac{r_2}{d_n}, \overline{\mathfrak{J}}\right) \end{cases}$$

Thus we shall have

$$\Delta_1\, \Upsilon_i = 4\,\pi\, \overline{\mathfrak{J}}\, d_1\, \mathfrak{n}_1$$
$$\Delta_2\, \Upsilon_i = 4\,\pi\, \overline{\mathfrak{J}}\, d_2\, \mathfrak{n}_2$$
$$. \quad . \quad . \quad . \quad . \quad .$$
$$\Delta_n\, \Upsilon_i = 4\,\pi\, \overline{\mathfrak{J}}\, d_n\, \mathfrak{n}_n$$

And by addition

$$(24) \quad . \quad . \quad . \quad \Sigma\, \Delta\, \Upsilon_i = 4\,\pi\, \overline{\mathfrak{J}}\, \Sigma\, d\, \mathfrak{n}$$

This last expression for $\Sigma \Delta \Upsilon_i$ must now be substituted for

the left-hand member in (IV), so that we obtain, after a simple transformation,

$$\text{(V)} \quad . \quad . \quad . \quad \bar{N} = \frac{2 \Sigma d \mathfrak{n}}{r_1 - \dfrac{\Sigma d}{2 \pi}}$$

In this expression for the mean value of the factor of demagnetisation of a multiply-divided toroid, the numerator (as in the corresponding equation (III) of § 80) depends solely on the form of the separate slits as such, while the denominator is proportional to the remainder of the circumference.

If we wish to determine the limiting value $\bar{N}_\infty$ of this factor, to which we found a first approximation in § 76, we have only to replace $\mathfrak{n}$ in (V) by the bracketed expression in equation (18), § 80, which gives us the equation

$$\text{(VI)} \quad . \quad . \quad \bar{N}_\infty = \frac{2 \Sigma \left(d + r_2 - \sqrt{d^2 + r_2^2}\right)}{r_1 - \dfrac{\Sigma d}{2 \pi}}$$

corresponding to the expression (II), p. 114.

In this chapter, however, we shall confine our attention entirely to toroids which are radially divided in one place only. (Compare also § 100.)

§ 82. **The Functions ν and $\mathfrak{n}$ are, approximately, Reciprocals.**—A simple relation, which is very approximately true, may be shown to exist between the function $\mathfrak{n}\,(r_2/d, \bar{\mathfrak{J}})$ already introduced (§ 80), and the leakage-coefficient ν previously defined.

In § 78, equation (17), we gave the definition

$$\nu = \frac{\bar{\mathfrak{G}}'_i}{\mathfrak{G}_i}$$

Or, remembering that the cross-section of the slit is the same as that of the rest of the toroid,

$$\text{(25)} \quad . \quad \nu = \frac{\bar{\mathfrak{B}}'_i}{\mathfrak{B}_i} = \frac{4 \pi \bar{\mathfrak{J}} + \bar{\mathfrak{H}}'_i}{\mathfrak{H}_i} = \frac{4 \pi \bar{\mathfrak{J}} + \bar{\mathfrak{H}}'_i}{\mathfrak{H}_i + \mathfrak{H}_e}$$

where the vectors in the denominator belong to the magnetically indifferent medium within the slits.

In the next place we may find a more simple expression for $\mathfrak{n}$, if we remember that $\mathfrak{H}_i$ alters very little as we pass along from one face of a slit to the other, so that the potential of this vector can be approximately represented by the product of its numerical value into the distance in question—that is, the width d of the slit. Hence we may write

$$ {}^{E}_{A}\Upsilon_i = \mathfrak{H}_i\, d $$

Substituting this in equation (19), § 80, we obtain

$$ (26) \qquad . \quad . \quad . \qquad \mathfrak{n} = \frac{\mathfrak{H}_i}{4\,\pi\,\mathfrak{J}} $$

On comparing now the expressions (25) and (26) for ν and $\mathfrak{n}$ respectively, we see that in both numerator and denominator of (25) the second term will be small compared with the first, provided we may assume that, as is nearly always the case, the external magnetising field is of only moderate intensity. Neglecting these second terms, we have

$$ (27) \qquad . \quad . \quad . \qquad \nu = \frac{4\,\pi\,\mathfrak{J}}{\mathfrak{H}_i} = \frac{1}{\mathfrak{n}} $$

so that to this degree of approximation ν and the function $\mathfrak{n}$ are reciprocal numbers.

An interesting case is that in which the slit is very narrow; if its width is made to diminish indefinitely, the leakage becomes less and less, and the coefficient ν which characterises it continually diminishes so as to approach the value unity. Its reciprocal $\mathfrak{n}$ at the same time increases so as to approximate to the same value, and in the limiting case $\nu = \mathfrak{n} = 1$, d being very small compared with r_1, so that equation (III), § 80 may be written in the following simplified form :—

$$ (\mathrm{VII}) \qquad . \quad . \quad . \quad . \; N = \frac{2\,d}{r_1} \quad . $$

When d/r_2 and d/r_1 are made indefinitely small in (II), which was originally obtained as only strictly true for an infinitely strong magnetising field, the equation becomes changed into the same simple form. The equation (VII) thus

holds good throughout, provided the slit is narrow enough to render the leakage negligible, and gives accordingly a factor of demagnetisation N which is constant throughout the whole range of magnetisation, as well as along the entire circumference of the ring, so that there is no need to introduce a mean value $\overline{N}$. If the width of the slit, instead of being given directly, is expressed as a percentage p of the circumference, or in angular measure a (degrees), equation (VII) may be written with practically sufficient accuracy in the following form[1]:—

$$\text{(VII}a\text{)} \quad . \quad . \quad N = \frac{1}{8}p, \text{ or } N = 0{\cdot}035\,a$$

B. *Experimental Tests*

§ 83. **The Iron Toroid Investigated.**—H. Lehmann has recently published an account of experiments conducted with the object of testing the author's theory of the magnetisation of a radially divided toroid.[2] We shall here describe these researches in some detail, in the first place because they constitute a safe experimental foundation for the physical laws of the magnetic circuit developed in this book; in the second place because they furnish a convenient example for the application of formulæ already given, as well as of Kirchhoff's method for determining normal curves of magnetisation; and finally because we cannot as yet assume that they are generally known, as we were able to do in the case of the fundamental researches quoted in the first chapter.

A complete toroid was turned from a plate of Swedish iron, and was afterwards carefully annealed. Its dimensions were then determined (partly by direct measurement, and partly by weighing) as follows (compare fig. 15, p. 109):—

Radius of the circular axis	. .	$r_1 = 7{\cdot}96$ cm.
Radius of cross-section	. .	$r_2 = 0{\cdot}895$ cm.
Area of cross-section	. . .	$S = 2{\cdot}52$ cm.^{2}.

[1] It was given by the author in this form (*Verh. physik. Ges. Berlin*, vol. **9**, p. 84, 1890; *Verh. der Sekt. Sitz. des Elektrotechn. Kongr. Frankf.* p. 73, 1891). More recently the problem of a divided toroid has been differently treated by Wassmuth, by the method of superposition and expansion in series described in § 71 (*Wien. Ber.* vol. **102**, part 2, p. 81, 1893).

[2] H. Lehmann, *Wied. Ann.* vol. **48**, p. 406, 1893.

After these measurements had been effected, the toroid was wound with wire. In order to leave sufficient room for the air-gap, two brass cheeks, b_1 b_2, were fixed on the iron at a distance of about 1 cm. apart, the winding being thus limited to the remainder of the circumference. In a position diametrically opposite to the middle of the arc b_1 b_2 a third cheek b_3 was fixed, so that it was possible to wind the two halves of the toroid separately. The general arrangement is shown in fig. 18.

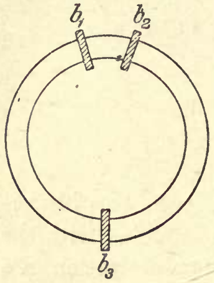

FIG. 18

The primary winding consisted of three layers of insulated copper wire 0·15 cm. thick, the total number of turns (n_1) being 695 (resistance = 0·51 ohm).[1] Over this again was wound a layer of thinner secondary wire, the number of turns (n_2) being 613, and the resistance 4·97 ohms. The magnetic intensity at the middle of the cross-section, arising from the current in the primary coil, was thus constant along the entire circumference, being given by equation (1) of § 72

$$\mathfrak{H}_e = \frac{2\,n_1\,I}{r_1} = \frac{2 \times 695}{7{\cdot}96} I = 174{\cdot}6\,I$$

Or, if I' denotes the strength of the current in ampères instead of in absolute units (deca-ampères),

$$\mathfrak{H}_e = 17{\cdot}46\,I'$$

The variations of $\mathfrak{H}_e$ from point to point of the cross-section may be neglected.

§ 84. **Standardisation of the Ballistic Galvanometer.**—To measure the electromotive impulse induced in the secondary coil, a ballistic galvanometer was used, whose complete period was 12 seconds. The throw, as is well known, is proportional to the quantity of electricity which passes through the galvanometer coils, and therefore also to the variations in the total flux of induction embraced by the secondary circuit.

[1] Of these 695 turns, 9 were between the cheeks b_1 and b_2, so that the external magnetising field could be properly considered to have a uniform peripheral distribution.

To determine the constant of the ballistic galvanometer, it was from time to time calibrated by means of a long, straight auxiliary coil, which had no iron core. The length (L) of this was 48·7 cm.; the mean diameter of the windings 3·552 cm., corresponding to a sectional area of 9·909 cm.^{2}; the number of turns (n) was 298, and the resistance 0·33 ohm. On this auxiliary coil a bobbin could slide, which carried 632 turns of secondary wire; its length was 5·7 cm., so that it extended over only a small fraction of the length of the auxiliary coil. Through this auxiliary coil let the current I'_A be sent, its value being measured by a reliable absolute ampèremeter. Then near the middle of the coil the magnetic field is nearly uniform, and [§ 6, equation (7)] is equal to

$$\mathfrak{H}_e = \frac{4\,\pi\, n\, I'_A}{10\, L}$$

Substituting in this expression the dimensions just given, we obtain

$$\mathfrak{H}_e = \frac{4\,\pi \times 298}{10 \times 48{\cdot}7} \times I'_A = 7{\cdot}69\, I'_A$$

The flux of induction $\mathfrak{G}_e$ due to the current is, in the absence of a ferromagnetic core, equal to the product of the intensity of field into the cross-section of the auxiliary coil. Hence,

$$\mathfrak{G}_e = \mathfrak{H}_e S = 7{\cdot}69 \times 9{\cdot}909\, I'_A = 76{\cdot}2\, I'_A$$

Thus, we have always the means of reproducing a flux of induction whose absolute value is known, so that, taking account of the total resistance of the secondary auxiliary circuit, the ballistic galvanometer can be standardised as often as may be required, and the proportionality between the throw and the value of $\mathfrak{G}_e$ controlled.

§ 85. **Tracing the Normal Curve of Magnetisation.**—Currents had to be sent through the primary wire having values up to about 20 ampères, so that the strength which was attained by the magnetic field was about 17·46 × 20, or, roughly, 350 C.G.S. units. The result is that in the comparatively thin wire considerable heat was developed, so that unless special precautions were taken the whole apparatus would be raised to an incon-

veniently high temperature. The toroid was therefore placed in a circular glass tank filled with pure petroleum, and was supported at a certain height above the bottom by wooden wedges. A block of ice was placed in the middle of the toroid, and served to keep the petroleum cool; while the water which accumulated from the melting of the ice was drawn off by a siphon. This simple device was found to answer perfectly.

The primary wire of the toroid was placed in circuit with a battery of accumulators, a commutator, a reliable ampèremeter, as well as rheostats, both metallic and liquid, by means of which the current could be brought either suddenly or gradually to any value up to 20 ampères. Before the commencement of each series of measurements the iron was demagnetised as far as possible by continually and rapidly reversing the commutator, while at the same time the adjustable fluid resistance was slowly increased, so that there was an exciting field rapidly alternating in direction, and gradually diminishing in intensity. Experience shows that by means of this method it is only possible to approximate more or less closely to the condition of the ferromagnetic substance previous to its first magnetisation. For complete demagnetisation, heating and subsequent slow cooling would be necessary; but, generally speaking, such an operation is of course impracticable.

The so-called 'curves of ascending reversals' (*aufsteigende Kommutirungskurven*) were determined by proceeding in a somewhat analogous manner; that is, by passing gradually from weaker to stronger values of current (and therefore of magnetising field), while for each value of the current (or of intensity of field) the direction of the current was reversed. The corresponding magnetisation was found from half the throw of the ballistic galvanometer as follows. Half the throw in question was multiplied by the constant of the galvanometer (which was determined in the manner mentioned above), and also by the total resistance of the secondary circuit. On dividing the product so obtained by 613, the number of turns in the secondary coil, we have the flux of induction $\mathfrak{G}'_i$ in the toroid. Let the quotient then be further divided by the cross-sectional area $S = 2{\cdot}52$ cm.^{2}, and the result is the resultant induction $\mathfrak{B}'_i$. Subtracting from this

$\mathfrak{H}'_e$ (which, in the case of a closed toroid, is identical with $\mathfrak{H}'_t$), and then dividing by 4π, the magnetisation [1]

$$\mathfrak{J} = \frac{\mathfrak{B}'_t - \mathfrak{H}'_e}{4\pi}$$

In this manner a determination was made of twenty-five points on the normal curve of magnetisation $\mathfrak{J} = \text{funct.}(\mathfrak{H}_e)$ (§ 13), each being deduced as a mean from ten readings [see Table II. and fig. 21, p. 131, curve (0)]. The curve so obtained characterises the specimen of iron examined, and constitutes a basis for the subsequent investigation.

TABLE II.—CURVE OF ASCENDING REVERSALS FOR A CLOSED TOROID (0)

$\mathfrak{H}_e$	$\mathfrak{J}$	$\mathfrak{H}_e$	$\mathfrak{J}$
0·6	93	18·1	1165
1·1	278	21·8	1192
1·8	440	27·5	1225
2·2	520	41·0	1275
3·2	690	55·1	1310
4·4	801	76·5	1348
5·3	871	104·5	1388
6·5	932	151·0	1440
7·5	985	200·0	1389
8·7	1020	240·0	1520
9·7	1051	315·0	1564
10·8	1076	385·0	1600
13·8	1118		

§ 86. **Arrangement of the Slit.**—After the normal curve of magnetisation had been duly determined, the continuity of the toroid was interrupted. By means of a circular saw about 0·1 cm. thick a radial cut of corresponding width was made in the place left free for the purpose between the two cheeks b_1 and b_2 (fig. 18, p. 124). In sawing, care was taken to leave the faces of the slit as plane and parallel and the edges as clean and sharp as possible. In order to render it possible to work with a narrower gap, a flexible brass hoop was fixed round the toroid, which could thus be bent together more or less closely by means

[1] For various small corrections, which must be introduced into this formula for a closed or divided toroid, as well as for other experimental details, we may refer to pp. 420, 434 of Lehmann's work already cited.

of the screw-clamp shown in fig. 19. A small disc, of magnetically indifferent brass, and of known thickness, was placed between the two faces, and then the toroid was bent together as far as this disc would allow (fig. 19).[1]

Further measurements were made with wider gaps, which were cut out by saws about 0·20 and 0·35 cm. in thickness respectively. In each case a special brass disc was fixed between the cut ends, so that the two faces bounding the air-gap could be held at a determinate distance apart. The rim of the little disc (which had been turned in a lathe) was then over-wound with several turns of very fine copper wire, so as to form a small secondary coil completely filling the gap. Care was taken to make the mean diameter of these windings equal to

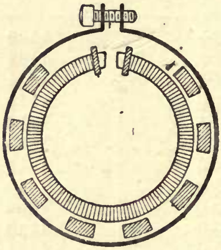

FIG. 19

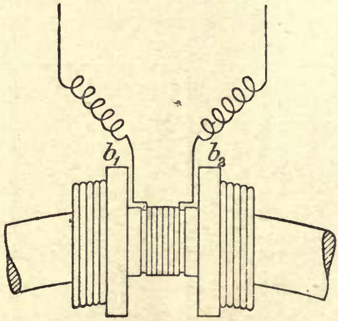

FIG. 20

that of the cut faces of the toroid, so that the current impulse induced in them directly measured the flux of induction $\mathfrak{G}_t$ through the gap. The ends of this small auxiliary coil were carefully insulated, and then the portion of the toroid between the cheeks b_1 and b_2 (fig. 20) was on each occasion re-wound with nine turns of the primary wire, so that the magnetic field produced had, as before, a uniform peripheral distribution.

The width of the gap was taken to be the mean between the thickness of the little separating disc, as determined by a micrometer, and the distance between the cut ends of the toroid,

[1] The slight distribution of stress thus produced in the ferromagnetic substance has so small an influence on its magnetisation that the effect may certainly be neglected. (Compare § 167.)

as measured by means of a dividing engine. The latter was, for obvious reasons, always somewhat in excess of the former.

Finally, it should be mentioned that, in order to determine the flux of induction through any given cross-section of the toroid, a movable secondary coil of seven turns was introduced, which embraced the toroid, and could be moved along the circumference from one part to another. It should be remarked that the large secondary coil surrounding the whole toroid measured the mean flux of induction $\bar{\mathfrak{G}}_t'$ in the iron, while the small auxiliary coil, wound on the rim of the separating disc, measured the flux $\mathfrak{G}_t$ through the gap. In accordance with the definition already given (§ 78), the former of these quantities, divided by the latter, gave directly the leakage-coefficient :

$$\nu = \frac{\bar{\mathfrak{G}}_t'}{\mathfrak{G}_t}$$

§ 87. **The Curves of Magnetisation.**—Lehmann's observations were made with five different widths of the gap :

No.	1	2	3	4	5
d:	0·040	0·063	0·103	0·202	0·357 cm.

and the corresponding results are set forth in five tables, of which we will here reproduce only one relating to measurements made with gap (3) ($d = 0{\cdot}103$ cm.). In the first column of Table III, p. 130, is given the intensity of field $\mathfrak{H}_e$ due to the primary coil; in the second the magnetisation $\mathfrak{I}$; in the third the mean flux of induction $\bar{\mathfrak{G}}_t'$ in the iron; in the fourth the flux $\mathfrak{G}_t$ through the gap. Then follow the leakage-coefficient $\nu = \bar{\mathfrak{G}}_t'/\mathfrak{G}_t$ in column 5, and its reciprocal $1/\nu = \mathfrak{G}_t/\bar{\mathfrak{G}}_t'$ in column 6. As we have already shown, the latter quantity is very nearly equal to the function $\mathfrak{n}$ introduced above.

We may be content to omit the other tables, since Lehmann has plotted curves to represent graphically the first two columns of each. To the right of the axis of ordinates (fig. 21, p. 131) the curves of magnetisation

$$\mathfrak{I} = \text{funct.}\ (\mathfrak{H}_e)$$

are drawn; first the normal curve (0) for the complete toroid, then the five other curves corresponding to different widths of

the gap. These are distinguished by the numbers 0, 1, 2, 3, 4, 5.

TABLE III

Gap (3). *Width of Gap* 0·103 *cm.*

$\mathfrak{H}_e$	$\mathfrak{I}$	$\mathfrak{B}_i'$	$\mathfrak{B}_i$	$\mathfrak{B}_i'/\mathfrak{B}_i = \nu$	$1/\nu\ \mathfrak{B}_i/\mathfrak{B}_i' =$
1·6	69	2197	—	—	—
2·4	117	3720	2045	1·82	0·550
3·8	206	6520	3660	1·78	0·561
5·1	279	8830	4880	1·81	0·553
6·8	383	12120	6720	1·81	0·553
8·9	492	15580	8610	1·81	0·553
11·9	628	19910	11230	1·78	0·563
14·9	748	23680	13400	1·77	0·565
18·0	852	26970	15250	1·77	0·565
20·9	923	29260	16650	1·76	0·568
24·3	988	31300	18000	1·74	0·575
27·2	1036	32850	18900	1·74	0·575
36·9	1141	36190	21400	1·69	0·591
49·0	1200	38120	23200	1·65	0·608
64·5	1250	39800	24750	1·61	0·621
78·5	1285	40870	25800	1·59	0·631
99·5	1325	42230	27030	1·56	0·641
181·0	1428	45600	30550	1·50	0·670
267·0	1500	48120	32800	1·47	0·681
(300)	1525	48900	32500	1·50	0·667

It must be noticed that for the abscissæ $\mathfrak{H}_e$ three different scales had to be used, so as to avoid unduly extending the height of the diagram (fig. 21). The first scale for abscissæ corresponds to ordinates ranging from $\mathfrak{I} = 0$ to $\mathfrak{I} = 1000$; the second scale, one-fifth of the former, from $\mathfrak{I} = 1000$ to $\mathfrak{I} = 1300$; and the third scale, one-twentieth of the first, from $\mathfrak{I} = 1300$ upwards. Thus, the inclinations of the curves to the axes of reference are throughout kept within convenient limits. The diagram shows at a glance how the influence of the gap gradually increases as its width is made greater.

§ 88. **Discussion of the Principal Results.**—The experimental data may be considered from three different points of view:—

I. *Lines of Demagnetisation*

If we suppose each of the curves 1, 2, 3, 4, 5 (fig. 21) to suffer separately a shear parallel to the axis of abscissæ until it is brought into coincidence with the normal curve (0), the

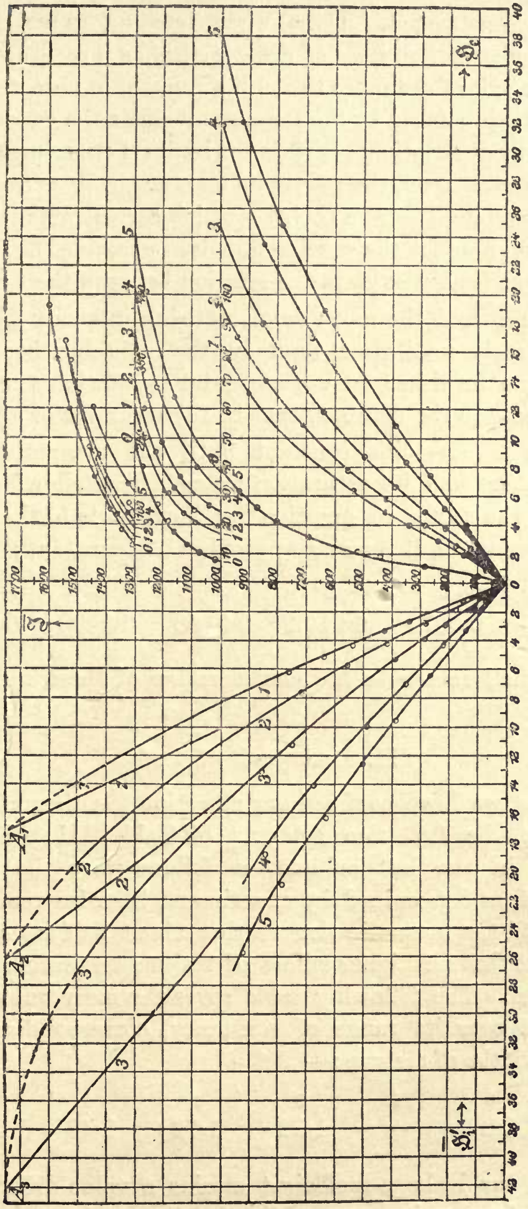

FIG. 21

directrix corresponding to each width of the gap employed will thus be constructed. These directrices, or, as we shall call them for distinction, *lines of demagnetisation*, are drawn to the left of the axis of ordinates through a number of observed points, and are distinguished by the same numbers as the corresponding curves of magnetisation. Within the limits of experimental error, as will be easily seen, the points lie on straight lines about as far as the ordinate $\mathfrak{I} = 875$ C.G.S. (which corresponds to half the saturation value for the specimen of iron examined). The lines of demagnetisation express the relation between the mean magnetisation $\overline{\mathfrak{I}}$ and the mean demagnetising intensity $\overline{\mathfrak{H}}'_i$. Their approximately rectilinear form in the neighbourhood of the origin thus furnishes an experimental proof that the ratio of the ordinate to the abscissa, that is, the mean factor of demagnetisation $\overline{N}$, remains constant until the magnetisation has reached about half its saturation value. The following are the values of the factor in question for those parts of the different curves over which it remains approximately constant:

No.	0	1	2	3	4	5
$\overline{N}$:	0	0·0079	0·0102	0·0140	0·0203	0·0246

We shall return to the consideration of these numbers in § 89.

II. *Leakage-Coefficients*

So far we have confined our attention to the numbers contained in the first two columns of Table III. From the numbers in the last column, it follows that the leakage-coefficient, as determined by experiment, likewise remains constant until the magnetisation reaches about half its saturation value, but that for higher values of the magnetisation it slowly diminishes. The following table gives the mean values of the coefficient over its range of constancy, corresponding to the various widths of air-gap:—

No.	0	1	2	3	4	5
ν:	1·00	—	1·52	1·79	2·48	3·81

Thus, the leakage-coefficient attains a value differing considerably from unity, even while the air-gap is still compara-

tively narrow. In fig. 22 the leakage-coefficient ν is plotted as a function of the mean magnetisation $\bar{\mathfrak{I}}$ (lower scale for abscissæ) for the gaps (2), (3), (4), (5); in the case of the narrowest gap, (1), no experiments on leakage were made. The initial constancy of ν here corresponds to the circumstance that the curves continue parallel to the axis of abscissæ until the magnetisation has reached about half its saturation value. As we pass to the higher values of the abscissæ, the curves begin to bend downwards. They are drawn as continuous lines as far as the experimental numbers extend, but if they are prolonged, as in the diagram, by the dotted parts, it appears that, as we approach the saturation point for the specimen of iron in question (about

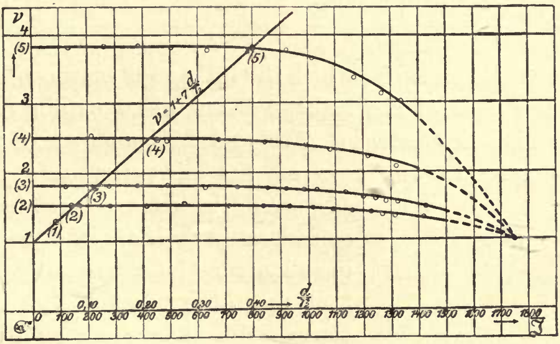

FIG. 22

1750 C.G.S.), they will all converge to the same point, whose ordinate corresponds to a leakage-coefficient equal to unity.

III. *Distribution of Leakage*

In the case of the three widest gaps employed, and for three values of the magnetisation (about 500, 1000 and 1500 C.G.S.), observations were made on the distribution of leakage along the circumference, using the movable auxiliary secondary coil mentioned in § 86. The results obtained are given in Table IV, p. 134, where the positions of the auxiliary coil on the circumference are entered as points of the compass, together with the corresponding values obtained for the flux of induction through this coil. From this table it follows that even for

TABLE IV

$\mathfrak{J}$	$\mathfrak{B}_S$	$\mathfrak{B}_{SE=SW}$	$\mathfrak{B}_{E=W}$	$\mathfrak{B}_{NE=NW}$	$\mathfrak{B}_N$
		Gap (3). $d = 0{\cdot}103$ *cm.*			
492	17100	16650	15850	14400	8610
988	33560	32850	32100	29400	18000
1525	48000	48000	48000	48000	32500
		Gap (4). $d = 0{\cdot}202$ *cm.*			
487	17630	17030	15950	13800	6230
997	34750	33550	32100	28700	13150
1520	46250	46000	46300	46100	24400
		Gap (5). $d = 0{\cdot}357$ *cm.*			
503	18800	17900	16200	13500	4160
1015	36200	34500	32700	28300	8800
1455	46350	46100	—	45800	16800

feeble magnetisations, for which the leakage is most considerable, the greater part of the leakage takes place within the short length which lies between NW and NE, and which contains the air-gap. In the case of the strongest magnetisation employed (the leakage-coefficient being considerably smaller), the property in question may be expressed by saying that up to the point NE or NW no considerable change occurs in the flux of induction through the cross-section of the toroid. The distribution of the induction is therefore sensibly uniform-peripheral over more than three-fourths of the circumference, and the uniformity of distribution will be the greater the higher the value which is reached by the induction.

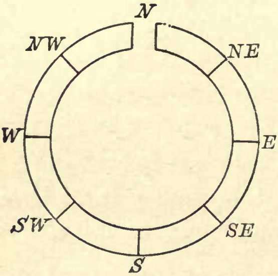

FIG. 23

§ 89. **Comparison of Theory and Experiment.**—We are now in a position to compare the results of the experiments described above with the conclusions of the theory previously developed. In equation (III) (§ 80)

$$\overline{N} = \frac{2\,d\,\mathfrak{n}\left(\frac{r_2}{d}, \mathfrak{J}\right)}{r_1 - \frac{d}{2\,\pi}}$$

we have a relation between the mean factor of demagnetisation $\overline{N}$ and the function $\mathfrak{n}$, which is the reciprocal of the leakage-coefficient ν. This last quantity, moreover, is represented graphically in fig. 22, p. 133, as a function of the magnetisation for the four widths of the gap (2, 3, 4, 5) which were employed. Again from (III) we easily obtain for the lines of demagnetisation the following equation:

$$(28) \quad . \quad . \quad . \quad \overline{\mathfrak{H}}_i = \frac{2\,d}{r_1 - \dfrac{d}{2\,\pi}} \cdot \frac{\overline{\mathfrak{I}}}{\nu}$$

which now enables us to construct the lines in question from the curves $\nu =$ funct. $(\overline{\mathfrak{I}})$ of fig. 22.

On the left-hand portion of fig. 21, p. 131, the lines ot demagnetisation constructed in this manner are shown. The lines (2) and (3) are continued as far as the ordinate $\mathfrak{I} = 1500$, because for higher values of $\mathfrak{I}$ the assumed reciprocity of $\mathfrak{n}$ and ν ceases to hold good with sufficient approximation (compare § 82). On the other hand (4) and (5) are only drawn for the range covered by the directly observed points.[1] As will be seen, these points lie approximately on the lines of demagnetisation. Thus, the theory leads to a satisfactory coincidence of the lines of demagnetisation plotted from measurements of the leakage with the curves of magnetisation which were determined by an entirely independent method.

Fig. 21 also gives for the three narrowest air-gaps, (1), (2), (3), between the values $\overline{\mathfrak{I}} = 1000$ and $\overline{\mathfrak{I}} = 1750$ C.G.S., the straight lines of demagnetisation whose equation is

$$(29) \quad . \quad . \quad . \quad \overline{\mathfrak{H}}_i = \overline{N}_\infty \, \mathfrak{I}$$

where $\overline{N}_\infty$ denotes that factor of demagnetisation which is to be found from equation II (§ 76), and which, in accordance with the assumption there made, is, strictly speaking, only applicable for infinitely high values of $\mathfrak{H}_e$.

From fig. 21, p. 131, it will now be observed how the values of $\overline{N}$, calculated from the measurements of leakage by means of

[1] Fig. 22 does not give the function $\nu =$ funct. $(\overline{\mathfrak{I}})$ for the gap (1), but, as we shall see in the next section, this can be found by interpolation. The line of demagnetisation (1) in fig. 21 was obtained in this manner.

equation III, and, as we have seen, according well with observation, tend also to the limit $\bar{N}_\infty$ as the saturation value $\mathfrak{J}_m = 1750$ C.G.S. is approached.[1] The lines of magnetisation may, in fact, be produced as in the dotted curves, so as to pass through the points A_1, A_2, A_3.

In Table V the somewhat complicated relations under discussion are collected in a form convenient for reference, so far as they correspond to the range of magnetisation $\mathfrak{J} = 0$ to $\mathfrak{J} = 875$, in which both the leakage-coefficient ν and the factor of demagnetisation $\bar{N}$ may be considered as constant. The meaning of each column will be sufficiently clear from its heading.

TABLE V

No.	d	$\frac{d}{r_2}$	ν	$\mathfrak{n}$	Calculated $\bar{N}_\infty$	Calculated $\bar{N}$	$\bar{N}$ (observed)	Percentage difference
1	0·040	0·045	1·31 [2]	0·765	0·0098	0·0077	0·0079	+ 2·5
2	0·063	0·070	1·52	0·660	0·0151	0·0105	0·0102	− 3
3	0·103	0·115	1·79	0·558	0·0242	0·0145	0·0140	− 3
4	0·202	0·226	2·48	0·403	0·0451	0·0205	0·0203	− 1
5	0·357	0·400	3·81	0 262	0·0726	0·0236	0·0246	+ 4

The agreement between the calculated and the observed values of $\bar{N}$ is as good as could be expected, when we remember that, on the one hand, the theory deals only with mean values and approximations, and that, on the other hand, the sources of experimental error, especially in relation to the exact form of the gap, may easily introduce an uncertainty of several per cents.

Finally, then, we may consider the theory here developed as confirmed by experiment with sufficient accuracy for most purposes; while the experimental data furnish us with the means of determining the function $\mathfrak{n}$ or its reciprocal ν, the

[1] To express the condition that magnetisation of the body is to be near the point of saturation, so that Kirchhoff's law of saturation becomes approximately applicable, it was supposed in § 57 that $\bar{\mathfrak{H}}_i'$ was small in comparison with $\mathfrak{H}_e$ (compare Culmann, *Wied. Ann.* vol. **48**, p. 380, 1893). From fig. 21 it will be seen that in reality for $\bar{\mathfrak{J}} = 1500$ C.G.S., $\mathfrak{H}_e$ was of an order of magnitude about tenfold as great as $\bar{\mathfrak{H}}_i'$.

[2] Value found by interpolation; compare the following section.

leakage-coefficient. These two quantities had to be provisionally introduced into our theory as unknown (§§ 78, 80).

§ 90. **Empirical Formula for the Leakage.**—On introducing the function $\mathfrak{n}$, we denoted it (§ 80) symbolically by

$$\mathfrak{n}\left(\frac{r_2}{d}, \quad \mathfrak{J}\right)$$

which expresses the fact that it depends only on the ratio r_2/d determined by the shape of the gap, and not on the radius of the entire toroid. In the following discussion we shall confine our attention to the range of magnetisation $\mathfrak{J} = 0$ to $\mathfrak{J} = 875$, which for practical applications is the most important; we can then put $\mathfrak{n} = 1/\nu$, and consider both these quantities to be independent of the magnetisation.

The question then arises how the leakage-coefficient ν (or the function $\mathfrak{n}$) depends on the shape of the air-gap, as determined by the ratio d/r_2 (or r_2/d). In order to obtain the experimental answer to this question, the ordinates ν in fig. 22, p. 133, are also plotted as a function of d/r_2; the second (upper) scale for abscissæ is introduced so as to allow this relation to be read off. We thus arrive at the empirical rule that, within the limits of experimental error, the four observed points lie on a straight line. This line cuts the axis of ordinates for the value $\nu = 1$, the corresponding law being that, when the width of the gap is reduced to zero, the leakage vanishes. The equation to the straight line is empirically found to be

$$(30) \quad . \qquad \nu = 1 + 7\frac{d}{r_2} \qquad \left[0 < \frac{d}{r_2} < \frac{1}{2}\right]$$

which may also be transformed as follows:

$$(31) \quad . \qquad \mathfrak{n} = \frac{\frac{r_2}{d}}{7 + \frac{r_2}{d}} \qquad \left[\infty > \frac{r_2}{d} > 2\right]$$

This formula for $\mathfrak{n} =$ funct. (r_2/d) is represented graphically by an hyperbola. In the square brackets following equations (30) and (31) is given the range of the independent variable within which the corresponding formulæ hold good. Too much weight must not be attached to such purely empirical relations

as these. For our physical insight into the phenomena they are quite useless; but, on the other hand, in practical applications it is useful to have at least some method for roughly estimating leakage-coefficients. We shall return to the consideration of this empirical formula for the leakage in § 173. In the present case, the formula can be applied to find by interpolation the value of the leakage-coefficient $\nu = 1{\cdot}31$ for the gap (1), for which it was not directly measured. This has already been done in Table V.

From the researches here described, it is clearly established that in a radially divided toroid the leakage for moderate values of the magnetisation $(0 < \mathfrak{J} < 875)$ remains nearly constant, while beyond this range the leakage decreases with increasing magnetisation. It also follows from our theory that this must be the case.

To show this, let us consider once more figs. 13, p. 87, and 17, p. 115. The acute angle $(90 - a')$, which the lines of induction within the toroid make with its bounding-surface, will become still more acute as the magnetisation increases beyond a certain value, since the peripherally directed distribution of the magnetisation (§ 76), and therefore also of the induction, tends to become more completely established in accordance with Kirchhoff's law of saturation. Let us further consider the tangent law of refraction for lines of induction, in accordance with which

$$\tan a = \frac{1}{\mu} \tan a'$$

We have just seen that, as the magnetisation increases, a' becomes greater, while, on the other hand, the permeability becomes smaller (§ 14); that is, $1/\mu$ becomes greater. These two causes conspire to increase the value of a, so that the lines of induction in the external medium will deviate further and further from the normal to the bounding-surface, the leakage thus becoming smaller, as was found by experiment to be actually the case.

Part II

APPLICATIONS

CHAPTER VI

GENERAL PROPERTIES OF MAGNETIC CIRCUITS

A. *Non-uniformly Magnetised Ring*

§ 91. **General Remarks.**—In the preceding chapter we have treated as rigorously as possible the typical case of a uniformly wound radially divided toroid, though only with a certain degree of approximation, and we have found our theoretical conclusions sufficiently confirmed by experiment. We shall now turn to cases in which the coils, or the ferromagnetic substance, are of a form less simple than that which has hitherto been assumed for the sake of theory, but which does not in general suffice for the applications of electromagnetism. In this, however, we may and must be content with rough approximations, for a rigorous mathematical treatment of such problems is as impracticable as it is useless.

If we have hitherto attempted to treat the questions which arise from the purely scientific point of view, we must now realise that it would be illusory to pursue that method any further, as it seems almost wholly out of the question to obtain in this way results of practical utility.

The new standpoint to which we refer is that of applied physics. The magnetisation $\mathfrak{J}$ at each point of the ferromagnetic substance, which from the physical point of view is the vector of most fundamental importance, is now no longer to be regarded as of the first consequence (compare § 12). We shall now rather concern ourselves with the induction $\mathfrak{B}$ or the flux of induction $\mathfrak{G}$; the latter vector is of the greatest practical importance, since its variations determine the electromotive forces induced in a conductor embracing a bundle of lines of induction.

§ 92. **Experiments of Oberbeck with Local Coils.**—We first attack the case of a closed toroid, which, however, is not now under the influence of a peripherally uniform magnetic field. Instead of the uniform winding hitherto assumed, there will only be one coil extending over a portion of the circumference, at S for instance. We shall first give the experimental investigation of this special case which has been made by Oberbeck.

The toroid of soft iron which he used had a mean radius $r_1 = 9{\cdot}5$ cm., the radius r_2 of the cross-section being 1 cm.[1] (fig. 15, p. 109). A primary coil of 145 turns was wound on it, and occupied one-fifteenth part of the circumference, so that the angle a (fig. 24) was 24°. A secondary coil of a few turns

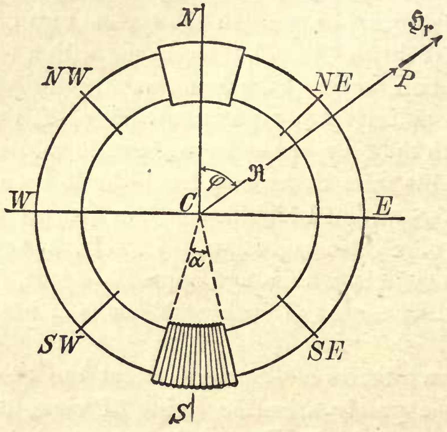

FIG. 24

could be moved over the whole circumference, and clamped in any given position. By reversing the magnetising current, a momentary current was produced in this secondary coil, which could be determined by means of a ballistic galvanometer, and this measured the flux of induction $\mathfrak{G}$ through the secondary coil, in the position which it then occupied. We again designate the position of the secondary coil on the circumference in fig. 24 by points of the compass as we did in

[1] Oberbeck, *Fortpflanzung der magnetischen Induktion im Eisen, Habilit. Schrift*, Halle, 1878. The statement of dimensions on p. 5 of that paper does not seem quite clear.

fig. 23, p. 134, and distinguish the corresponding values of the flux of induction by suffixes as in Table IV, p. 134. We start from the highest value of the flux of induction, which is obviously at S, and therefore put $\mathfrak{G}_S = 100$. Oberbeck found in one case, for example,

$$\mathfrak{G}_E = \mathfrak{G}_W = 93$$

and

$$\mathfrak{G}_N = 91$$

Having regard to the very unequal values of the magnetising field of the coil at different parts of the circumference of the toroid, the flux of induction may be said to be almost constant, especially over that half of the toroid WNE which is not overwound. From the fall in the value of $\mathfrak{G}$, towards that part which is directly opposite the coil, it follows that some induction-tubes (§ 61) must emerge from the ferromagnetic substance, and spread out into the surrounding indifferent medium; in other words, there is a leakage, which manifests itself by magnetic action at a distance (compare §§ 15, 78).

On using one coil at S and one at N, each of which occupied an angle of about 18°—that is, the twentieth part of the circumference—and each of which tended to produce magnetisation in the same direction, the result obtained was:

$$\mathfrak{G}_S = \mathfrak{G}_N = 100$$
$$\mathfrak{G}_E = \mathfrak{G}_W = \ \ 98$$

The variation was therefore extremely small. The case in which the magnetising coils acted in opposition to each other, whether they were of the same or of different strengths, was also examined by Oberbeck, but has less interest for our present purpose.

§ 93. **Further Experiments by von Ettingshausen and Mues.**— Soon after the publication of Oberbeck's investigations, von Ettingshausen[1] published measurements which were quite

[1] Von Ettingshausen, *Wied. Ann.* vol. **8**, p. 554, 1879. These experiments were made with the view of testing an equation which Boltzmann (*Wiener Anzeiger*, No. 22, p. 203, 1878; *Wiedemann's Beiblätter*, vol. **3**, p. 372, 1879) had deduced for the present case. This formula has a purely mathematical interest, as it is based on the assumption of constant susceptibility, which, under no circumstances, is accurate, and especially in the present case, as von Ettingshausen himself observes, does not appear capable of giving even approximately correct results.

similar. He first used a welded toroid, of ordinary bar iron ($r_1 = 12{\cdot}26$ cm., $r_2 = 0{\cdot}77$ cm.), in which the coil at S covered one forty-fifth of the circumference ($a = 8°$). Greater differences of the flux of induction between S and N were now observed than had been found by Oberbeck. These differences were, however, less the stronger was the primary current. After this a second toroid was turned from a plate of Styrian soft iron so as to have no weld ($r_1 = 10{\cdot}95$ cm., $r_2 = 0{\cdot}75$ cm.); it was wound in just the same way as Oberbeck had done—that is to say, with 145 turns on a fifteenth part of the circumference ($a = 24°$). The results agreed better with those of Oberbeck, and the more closely the stronger was the magnetising current.

In the experiments of von Ettingshausen the absolute system of measurements is adopted, which was not the case with those of Oberbeck. The latter probably employed stronger magnetising fields than the former (taking into consideration the mean value of the magnetic intensity which varies along the circumference). This is probably the origin of the difference in the results of the two investigators, as will be more completely discussed in the following paragraphs.

More recently, Mues,[1] under the guidance of Oberbeck, has investigated the case of a ring magnetised by two local coils in a somewhat different manner by measuring the action at a distance due to leakage. The annealed iron rings were rectangular in section, and were wound at two places diametrically opposite each other with coils which tended to magnetise the ring in the same sense and to the same extent. In determining the action at a distance only points in the plane of the ring (for instance, P, fig. 24, p. 142) were considered, and at these points the radial components of the field $\mathfrak{H}_r$ were especially determined.

From considerations of symmetry it is evident that these radial components must vanish at all points which lie in the straight lines $\overline{NS}$ and $\overline{WE}$,[2] and this was also verified by

[1] Louis Mues, *Magnetismus von Eisenringen*. Dissertation, Greifswald, 1893; also *Wiedemann Beiblätter*, vol. **18**, p. 592, 1894.

[2] It scarcely needs mention that the terms derived from the compass, especially N and S, have nothing to do with north and south magnetism, but only serve for orientation.

experiment. If the position of the point is defined firstly by its azimuth ϕ measured from $\overline{NS}$ (fig. 24, p. 142), and secondly by its distance $\mathfrak{R}$ from the centre C of the ring, then from the nature of the case the value of the radial component must be a periodic function of the azimuth ϕ, the period of which is π, and which can be expanded in a Fourier's series.

Now experiment showed that the first term of such a series, which of course is proportional to sin 2ϕ, represented the measurements with sufficient accuracy. The value of $\mathfrak{H}_r$ was a maximum therefore for $\phi = 45°, 135°, 225°, 315°$, while it was zero for $\phi = 0°, 90°, 180°, 270°$. If, for example, a radial component of magnetic force directed outwards from C resulted from a given direction of current in the NE quadrant, this was also the case in the SW quadrant, while on the contrary the radial components in the SE and NW quadrants would be directed towards the centre of the ring. It was finally established that within a certain range $\mathfrak{H}_r$ was inversely proportional to $\mathfrak{R}^4$.

§ 94. **Theoretical Explanation of the Experiments.**—These experiments of Oberbeck and of von Ettingshausen show that the total flux of induction $\mathfrak{G}_t$ along the circumference of the toroid is approximately constant; the section being constant, this holds also for the resultant induction $\mathfrak{B}_t$. This finally proves that the resultant magnetic intensity $\mathfrak{H}_t$ which produces that induction $\mathfrak{B}_t$ is likewise free from perceptible variations. Since[1] $\mathfrak{H}_t = \mathfrak{H}_e + \mathfrak{H}_i$ (in the algebraic sense, in accordance with (1) § 53), it follows finally from experiment, that the enormous variation of the member $\mathfrak{H}_e$, which is due to the magnetising coil, is almost compensated by that of the demagnetising force $\mathfrak{H}_i$, which also varies along the circumference; this vector is opposite in direction to the field of the coil, and, owing to the comparatively small values of the latter, it will also be of the same order of magnitude (compare §§ 18, 53, 54).

On the other hand, as the flux of induction is only approximately constant, it follows that a few tubes of induction will always emerge, and that, in consequence, magnetically effective end-elements will exist at the curved boundary of

[1] Concerning the meaning of the suffixes *e*, *i*, *t*, reference should be made to p. 80, where they are fully discussed.

the toroid, and the strength of these will determine the necessary compensating variations of the demagnetising force $\mathfrak{H}_i$. We finally come to what, at first sight, is a surprising result, namely, that just because the toroid is only magnetised by a local coil, and therefore a leakage of induction-tubes occurs, the magnetic end-elements thereby produced tend to compensate for the inequalities of the field of the coil. Consequently the distribution of the total intensity, and with it that of the magnetisation, as well as of the induction, will be little different from a uniform-peripheral one, which as a matter of fact is what takes place.

It is further intelligible why, in the experiments described, a uniform-peripheral distribution of the resultant induction is the more nearly reached the stronger is the magnetising current. For since, so far as we can judge, the maximum susceptibility was not attained with the current employed, this quantity must increase with the current. It is, however, obvious that, other things being equal, the greater the susceptibility, the more completely must the field arising from the magnetisation of the toroid compensate for the inequalities of the field which is due to the magnetising coil. This is in accordance with von Ettingshausen's observations, who found, with a welded toroid of ordinary bar iron, less complete uniformity than with one turned out of a Styrian soft iron plate, of undoubtedly far higher susceptibility.

§ 95. **Self-Compensating Effect of Leakage.**—This self-opposing action of leakage in magnetic circuits which we have observed in the comparatively simple case of the toroid, is a perfectly general phenomenon. Differences in the flux of induction at different parts of the circuit cause some tubes of induction to emerge—that is, leakage occurs, so that magnetic end-elements make their appearance, their strength being greater the more unit-tubes emerge. The direction of the magnetic intensity due to these elements will evidently be opposed to the external field in those places where the flux of induction is greatest, and in agreement with the external field in places where the flux is least. In this way the variations of the flux of induction at different places along the circumference are, in a certain measure, automatically diminished.

The self-compensating effect of leakage in magnetic circuits affords a certain analogy with the demagnetising tendency of the faces which bound the interspace. Both actions may, as follows, be brought under one point of view. We have seen (§ 50) that any apparent action at a distance is due to local variations in the 'strength' of the magnetisation. Now it may easily be shown that the magnetic action at a distance arising from these variations, is opposite in direction to the magnetisation at places where the strength is greater, while at places of less strength it is in the same direction; the local variations will therefore indirectly tend to partially neutralise themselves.

In the experiments which have been described the field was so weak as to satisfy the conditions of § 11 (equation 14), and therefore the magnetisation was proportional to the induction, and, like this vector, it had a solenoidal distribution. It is accordingly sufficient to assume magnetic end-elements on the lateral surface as the direct result of leakage. The theoretical formulation last mentioned includes also, such variations in the strength, as take place in the interior of the ferromagnetic substance when the distribution of the magnetisation is no longer solenoidal, so that its convergence is finite, as may readily occur with higher intensities of field (compare §§ 11, 59).

The leakage in the latter case cannot, in general, be determined, nor can we fix the limiting conditions corresponding to a magnetising field whose intensity increases without limit. In any case, however, Kirchhoff's law of saturation will then hold. The question, then, is how the lines of magnetic intensity of the external field would run in reference to the geometrical configuration of the magnetic circuit; for in the limiting case, in accordance with the principle in question, these alone will completely determine and direct the other vectors concerned. The answer to this question depends on the nature of the special case we consider, and will usually present considerable difficulties. Even in the simple case of a toroid magnetised by a local coil covering one special part, it depends on the relation of the dimensions of the coil to the diameter of the toroid;[1] so that any general enun-

[1] Compare in this connection the graphical representation of the lines of intensity of a single circular conductor, Maxwell, *Treatise*, vol. 2, Plate XVIII.

ciation, not to speak of a solution, of the problem, applicable to all cases, seems out of the question.

Finally, since it has been experimentally established and theoretically explained how the value of the vector $\mathfrak{H}_i$ is sensibly constant throughout a ring magnetised by a local coil, the question arises how the mean value of that vector can be calculated. To this end, we observe that the line-integral of $\mathfrak{H}_e$ along a path of integration lying within the toroid is, as before, $4\pi n I$, in which n is the number of turns of the local coil, I the current in deca-ampères (*i.e.* in absolute C.G.S. measure). The line-integral of $\mathfrak{H}_i$ is, however, zero in this as in all other cases. The mean value $\bar{\mathfrak{H}}_i$ is found by dividing the mean circumference of the toroid $(2\pi r_1)$ into the sum of the two line-integrals just mentioned, which sum, in this case, evidently also amounts to $4\pi n I$. Accordingly,

$$(1) \qquad \bar{\mathfrak{H}}_i = \frac{4\pi n I}{2\pi r_1} = \frac{2 n I}{r_1}$$

the same expression which we previously found for the case of a uniform winding [§ 72, equation (1)], that is for a uniform peripheral distribution of the magnetising field. We shall consequently drop, in the sequel, the tacit or express assumption that the winding is uniform.

B. *Hopkinson's Synthetic Method*

§ 96. **Principles of the Method.**—We now turn to a mode of treating the magnetic circuit, as ingenious as it is fruitful, which was published in 1886 by Drs. J. and E. Hopkinson.[1] It depends on two fundamental ideas, each of which, again, is based on a mathematical theorem capable of rigorous proof.

The starting-point is the consideration of the total flux of induction, the 'conservation' of which is the first of the principles referred to. This we have already fully discussed—or, what amounts to the same thing, we have established the general solenoidal property of the resultant induction, as

[1] J. and E. Hopkinson, *Phil. Trans.* vol. **177**, I. p. 331, 1886. Reprinted in J. Hopkinson, *Original Papers on Dynamo Machinery and Allied Subjects*, p. 79, New York 1893.

expressed by the equation of continuity, and have shown how this principle governs a series of phenomena (§§ 60–65).

In the second place, the fundamental principle is applied that the line-integral of the resultant magnetic intensity $\mathfrak{H}_t$ along any closed curve is $4\pi n I$, where n is the number of turns of the conductor which embrace the curve, I the current which passes through all of them (§ 56).

The magnetic circuit is separated into its natural parts, through which the curve of integration successively passes. To each such part of the circuit corresponds a portion of the line-integral in question, the integral being calculated by multiplying the mean value $\overline{\mathfrak{H}_t}$ for each part by the corresponding length of the path of integration. An essential assumption here tacitly made is the tendency of the resultant induction $\mathfrak{B}_t$, already discussed, to become distributed as uniformly as possible, so that its variations along any one such portion of the curve are but small. From $\mathfrak{B}_t$ we find $\mathfrak{H}_t$ by equation (3*a*) p. 147.

The several portions of the integral are finally added, and their sum must amount to $4\pi n I$. We can thus ascertain synthetically the value of the line-integral corresponding to any given or prescribed value of the total flux of induction $\mathfrak{G}_t$, which we will call M. The relation between these two quantities M and $\mathfrak{G}_t$ we can represent graphically; and, according to the practice of the Drs. Hopkinson, it is usual to choose the values of the former as abscissæ and those of the latter as ordinates. The curve thus obtained, which represents the 'Hopkinson's function,'

$$(2) \qquad M = F_H(\mathfrak{G}_t) \quad \text{or} \quad \mathfrak{G}_t = \Phi_H(M)$$

may be called the *magnetic characteristic* of the corresponding magnetic circuit.

In order that the method, here represented in its general features, may be available, the relation between the vectors $\mathfrak{B}_t$ and $\mathfrak{H}_t$, which, as we know, are coincident in direction (§ 54) at each point, must be given for the ferromagnetic substance in question. We can represent it by the equation

$$(3) \qquad \mathfrak{B}_t = \phi(\mathfrak{H}_t)$$

or, conversely, by

$$(3a) \qquad \mathfrak{H} = f(\mathfrak{B}_t)$$

The functions f and ϕ, as well as F_H and Φ_H, are 'inverse functions.'[1] The former are assumed to be given empirically by the normal curve of induction for the material in question.

§ 97. **Application to Radially-divided Toroids.**—As it may at first prove somewhat difficult to understand Hopkinson's synthetical method, we will once more explain the mode of applying it in the typical case of a toroid with a radial slit. We shall then ultimately see that it leads to the same results as the method which we have developed in Chapter V., and which, at first sight, appears to be totally different.

Using the same notation as before (§ 75, fig. 15), we assume as a first approximation that the width of the slit d is small, so that the leakage may be neglected. Or, as the Drs. Hopkinson say, we suppose that, by some miracle, the tubes of induction are prevented from emerging from the surface of the toroid, so that they can only pass from one face to the other through the gap. Let S $[= \pi r_2^2]$ be the cross-section of the toroid, as well as of the slit; then, on the above assumption,[2]

$$\mathfrak{G}'_t = \mathfrak{G}_t = \mathfrak{B}_t S$$

Let us now evaluate the separate portions of the line-integral mentioned above. For this we first consider the slit, where $\mathfrak{B}_t = \mathfrak{H}_t$; consequently (see fig. 15, p. 109),

$$(4) \quad . \quad . \quad \int_A^E \mathfrak{H}_t \, dL = \mathfrak{H}_t \, d = \mathfrak{B}_t \, d = \frac{\mathfrak{G}_t}{S} d$$

Next, in the remaining ferromagnetic part of the toroid, by introducing the mean value of the resultant intensity $\overline{\mathfrak{H}}'_t$,

$$(5) \quad \int_E^A \mathfrak{H}'_t \, dL = \overline{\mathfrak{H}}'_t \, (2 \pi r_1 - d) = (2 \pi r_1 - d) \, f\left(\frac{\mathfrak{G}_t}{S}\right)$$

in which f is the function defined by equation (3a). The sum of the two portions (4) and (5) of the integral must, from the

[1] In Hopkinson's paper ϕ is put $= f^{-1}$ in conformity with the usual notation for inverse functions.

[2] It should be remembered that when a symbol is accented, it denotes the value of the corresponding quantity within the ferromagnetic substance.

preceding paragraph, amount to $M = 4\pi I n$. Hence we finally obtain

$$(\mathrm{I}) \qquad M = 4\pi n I = \frac{\mathfrak{G}_t}{S}\, d + (2\pi r_1 - d)\, f\left(\frac{\mathfrak{G}_t}{S}\right) = F_H(\mathfrak{G}_t)$$

This equation represents Hopkinson's solution of the problem of the radially-divided toroid.

§ 98. **Graphical Representation. Transformation of Curves.**—In fig. 25 this solution is graphically represented for a concrete case. The toroid is assumed to be of the specimen of iron whose normal curve of magnetisation is represented in fig. 5,

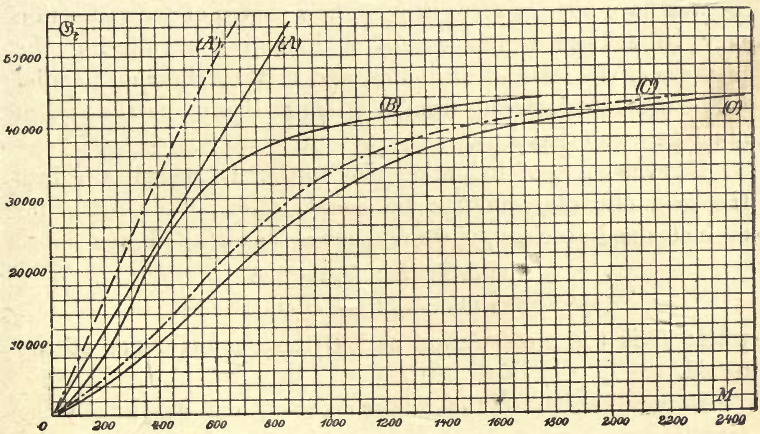

FIG. 25

p. 24, by the curve A. Let its dimensions in round numbers be as follows:

$r_1 = 10$ cm.	Circumference $L = 2\pi r_1 = 62{\cdot}83$ cm.
$r_2 = 1$ cm.	Cross-section $S = 2\pi r_2^2 = 3{\cdot}14$ cm².
$d = 0{\cdot}05$ cm.	Ratio $d/r_2 = 0{\cdot}05$.

The two terms of equation (I) corresponding to the two portions of the integral are now represented by curves (A) and (B), the first of which is obviously a straight line through the origin. If then we add the abscissæ of these curves (A) and (B), we obtain a third one (C), which represents the relation sought for between $\mathfrak{G}_t$ and M, and which is therefore the magnetic characteristic of the divided toroid.

It is at once evident that this addition of abscissæ amounts to the same thing as if we had sheared the curve (B) parallel to the axis of abscissæ, by starting from a directrix to the left of the axis of ordinates, symmetrically related to the straight line (A), to the right, and moving the former into coincidence with the axis of ordinates. This points directly to an analogy with the method followed in the last chapter (compare fig. 21, p. 131). In order to explain more clearly these analogies, we shall endeavour to transform the curve of magnetisation previously[1] given [$\mathfrak{J} = \text{funct.}\ (\mathfrak{H}_e)$] into the magnetic characteristic [$\mathfrak{G}_i = F_H\ (M)$] represented by curve ($C$), fig. 25.

We have already seen, in § 13, how curves of magnetisation may be converted into curves of induction by altering the measure of the scale of ordinates, and then by shearing parallel to the ordinates starting from a straight line under the axis of abscissæ up to that axis.

But the curve of induction may further be transformed into the magnetic characteristic by merely altering the scales for ordinates and abscissæ, if we remember that in the latter

$$\text{the ordinate: } \mathfrak{G}_i = \mathfrak{B}_i S$$
$$\text{the abscissa: } M = \mathfrak{H}_e L$$

where S is the cross-section and L the circumference of the toroid.

It may be left to the reader to carry out graphically the complete transformation of the curve (B), fig. 5, into curve (C), fig. 25, in the manner described. It will then be seen that the two entirely different modes of representation come to exactly the same thing, as the curves in question finally coincide.

§ 99. **Second Approximation. Correction for Leakage.**—In the further course of their investigation the Drs. Hopkinson drop the simplifying assumption that leakage is prevented, and introduce the coefficient of leakage ν, which we have already had occasion to employ. We have given as the equation defining it [§ 78, equation (17)]

$$(6) \quad . \quad . \quad . \qquad \nu = \frac{\mathfrak{G}'_i}{\mathfrak{G}_i} \qquad [\nu \geq 1]$$

[1] That is in fig. 5, p. 24, the curve (B) which not only as observed in the text refers to the same material, but is also drawn for the same dimensions as in the example chosen.

where $\overline{\mathfrak{G}}'_t$ is the mean total flux of induction in the ferromagnetic substance, $\mathfrak{G}_t$ that in the gap. The resultant induction here is

$$(7) \qquad \mathfrak{B}_t = \frac{\mathfrak{G}_t}{S} = \frac{\overline{\mathfrak{G}}'_t}{\nu S}$$

On the other hand, the mean value of this vector in the ferromagnetic substance is

$$(8) \qquad \overline{\mathfrak{B}}'_t = \frac{\overline{\mathfrak{G}}'_t}{S} = \frac{\nu \mathfrak{G}_t}{S}$$

If we take into account this modification due to leakage in the investigation of § 97, equation (I), as is readily seen, becomes to a second approximation

$$\text{(II)} \quad M = 4\pi n I = \frac{\mathfrak{G}_t}{S} d + (2\pi r_1 - d) f\left(\frac{\nu \mathfrak{G}_t}{S}\right) = F_H(\mathfrak{G}_t)$$

This is the fundamental equation of the Hopkinson method, which may be immediately extended to complicated cases, as we shall see in the following paragraphs. But if, for the sake of better comparison, we take as independent variable $\overline{\mathfrak{G}}'_t$ in the ferromagnetic substance instead of $\mathfrak{G}_t$ in the gap, we have

$$\text{(IIa)} \quad M = 4\pi n I = \frac{\overline{\mathfrak{G}}'_t}{\nu S} d + (2\pi r_1 - d) f\left(\frac{\overline{\mathfrak{G}}'_t}{S}\right) = F'_H(\overline{\mathfrak{G}}'_t)$$

If we also plot this latter function F'_H graphically, and for the same concrete case as the above, we see how, instead of the straight line (A), which represented the first term in equation (I), we have now the dotted line (A') (fig. 25, p. 151). The value of ν, in so far as it is constant, is taken from the empirical equation [§ 90, equation (30)]:

$$\nu = 1 + 7\frac{d}{r_2}$$

And hence, since $d/r_2 = 0{\cdot}05$, we shall have $\nu = 1{\cdot}35$. We know, further, that it more and more approaches the value unity the more nearly the ferromagnetic substance approaches the point of saturation.[1] The Hopkinson function F'_H is then

[1] The ultimate decrease of ν manifests itself in fig. 25 only by the fact that the original straight line (A') for values above $\mathfrak{G} = 40000$ is somewhat inclined to the straight line (A).

obtained by adding the abscissæ; it is now represented by the dotted curve (C').

The construction formerly given (§ 17), the analogy of which with the Hopkinson method was thoroughly discussed, undergoes quite similar modifications by taking leakage into account (fig. 21, p. 131), so that the agreement of the two methods is the same as before. We can therefore regard the confirmation of the author's theory (Chapter V.) by the experiments of Lehmann, made expressly with this view, also as a confirmation of the Hopkinson method—at any rate in so far as it deals with the simple case of a toroid with one radial slit. Of the two methods, which at first appear totally different, sometimes the one has the advantage, and sometimes the other, from the point of view of practical application.

§ 100. **Generalisation of the Method.**—The Hopkinson method thus has the advantage that it can be directly applied to imperfect magnetic circuits of a more general kind than the typical example previously considered. We have already treated theoretically the case of a toroid with several radial slits, taking leakage into account (§ 81). Introducing the generalisation of § 15, according to which the curved axis of the ring may be any arbitrary plane or tortuous curve, provided only its radius of curvature is always great compared with the greatest diameter of the normal cross-section, this latter may have any arbitrary though invariable form. If, finally, we remove these latter restrictions also, so that the curve may now be sharply bent, and the shape, as well as the area of the cross-section, may be variable, and if we further assume that the ferromagnetic parts consist of different materials, we have obviously an imperfect magnetic circuit of the most general kind possible.

We now start from the mean flux of induction $\mathfrak{G}_0$ in one of the gaps, the length and sectional area of which are L_0 and S_0 respectively. In the other parts, 1, 2, 3, &c., into which we can separate the magnetic circuit, let the values of the flux of induction be $\nu_1 \mathfrak{G}_0$, $\nu_2 \mathfrak{G}_0$, $\nu_3 \mathfrak{G}_0$, and so on. In like manner, let the particular functions f, which are characteristic of the various ferromagnetic substances forming those parts, be f_1, f_2, f_3, &c. [§ 96, equation ($3a$)]. If we call the corresponding lengths of path L_1, L_2, L_3, &c., and the sectional areas S_1, S_2, S_3, &c., we

ultimately obtain from equation (II), p. 153, the more general equation

$$\text{(III)}\quad M = 4\pi n I = L_0 \frac{\mathfrak{G}_0}{S_0} + L_1 f_1 \left(\frac{\nu_1 \mathfrak{G}_0}{S_1}\right) + L_2 f_2 \left(\frac{\nu_2 \mathfrak{G}_0}{S_2}\right) + \ldots = F_H(\mathfrak{G}_0)$$

In so far as, besides the parts of the magnetic circuit represented by the first term, there are others of indifferent material—for instance, the nth part—to each of these parts there will correspond a linear term

$$(9) \qquad . \qquad . \qquad . \qquad L_n \frac{\nu_n \mathfrak{G}_0}{S_n}$$

since in this case the function f_n is simply equal to its independent variable.

This most general equation has, so far, been scarcely tested by purely magnetic experiments. Any attempt at an accurate test would, moreover, be useless, as the method is intended for practical requirements, and, of course, only allows of mean values and rough approximations. We shall revert to this in Chapter VIII., in discussing the chief applications in connection with dynamos, and show how the Drs. Hopkinson succeeded in applying their determination of the electromotive force of the machine as a function of the current in the field-magnets to give a test of the theory. By their measurements the approximate correctness of the synthetical method could, on the whole, be confirmed (compare especially §§ 128, 129).

C. *Electromagnetic Stress*

§ 101. **Specification of the state of Stress.**—We will now consider more closely the state of stress which occurs in the ferromagnetic parts of magnetic circuits, or which prevails in the gaps between them. The latter explains the well-known apparent action at a distance, which appears, in general, as an attraction or repulsion between the ferromagnetic parts. To these considerations we will preface a few elementary definitions.

By *stress* is understood, in general, a system of forces which tend, not to move a body, but to strain it. The stress produces, in general, a *strain*. The closer investigation of the latter, or of its relations to the stress, constitutes a problem of geometry, or of the theory of elasticity respectively.

Every stress is to be expressed as a force per unit area, and has therefore, in absolute measure, the dimensions $[L^{-1}MT^{-2}]$. There are various elementary forms of stress; the most important are *shearing stress*, *pull* or *tension*, and *thrust* or *pressure*.

That being premised, we may mention that in the theoretical part (§ 65) we have already expressed in a few equations the most general state of stress in a magnetic body, as deduced mathematically by Maxwell. Making the simplification that the induction $\mathfrak{B}$ and the magnetic intensity[1] $\mathfrak{H}$ have the same direction, which, on the assumption always made of isotropy and absence of hysteresis (§ 54), does actually hold, those equations assume an elementary form; and, in accordance therewith, the stress may be completely specified in terms of the two following elementary forms:—

I. A (hydrostatic) pressure, the same in all directions, and equal in absolute measure to $\mathfrak{H}'^2/8\pi$.

II. A simple tension in the direction of the lines of induction, whose value in absolute measure is $\mathfrak{B}'\mathfrak{H}'/4\pi$.

The investigation of the strains[2] due to the electromagnetic stress concerns the theory of elasticity. The test whether such strain is admissible, from the point of view of construction, is a question of strength of materials. From neither of these points of view need we follow the question in this book; but its importance is apparent from what has been said, for, as we shall see, the stress may in some circumstances be very great.

§ 102. **Resultant Tension in the Gap.**—In the preceding paragraph the stress in a ferromagnetic body is fully specified. In the sequel we shall consider only that one of its manifestations which is of greatest experimental and practical interest; we

[1] Since, in the sequel we shall only deal with the resultant intensity, and the resultant induction, we shall drop the prefix 'resultant' as well as the corresponding index t from the corresponding symbols (§ 53).

[2] We have already alluded to stresses which occur in magnetisation, and to the small changes of size and shape of ferromagnetic bodies due (at any rate partly) to them.

shall confine ourselves to an investigation of the resultant tension in the direction of the lines of induction. This vector (in the ferromagnetic substance) we shall call $\mathfrak{Z}'$. We obtain its value if, from that of the simple tension in the direction of the lines of induction (2), we subtract the pressure mentioned in (1), which of course acts in the direction in question, as well as in all others. We accordingly obtain the equation

$$(10) \quad . \quad . \quad \mathfrak{Z}' = \frac{1}{4\pi}\mathfrak{B}'\mathfrak{H}' - \frac{1}{8\pi}\mathfrak{H}'^2$$

Let us now consider an infinitely narrow slit in the ferromagnetic substance at right angles to the direction of the lines of induction and intensity, which does not produce either demagnetising actions or leakage. As this will in practice be always more or less the case, we shall speak of such a break in the continuity of the ferromagnetic substance as an *ideal slit*. $\mathfrak{H}$ and $\mathfrak{B}$ are there identical, as in an indifferent substance, and are equal to $\mathfrak{B}'$ in the ferromagnetic substance, as follows from the principle of the normal continuity of induction (§ 58); hence the resultant tension $\mathfrak{Z}$ in the ideal slit itself is

$$(\mathrm{IV}) \qquad \mathfrak{Z} = \frac{1}{4\pi}\mathfrak{H}^2 - \frac{1}{8\pi}\mathfrak{H}^2 = \frac{1}{8\pi}\mathfrak{H}^2 = \frac{1}{8\pi}\mathfrak{B}'^2$$

This value for $\mathfrak{Z}$ in the narrow gap is considerably greater than that for $\mathfrak{Z}'$ in the ferromagnetic substance. We now take the difference of the two quantities and introduce the magnetisation, remembering the fundamental equation

$$\mathfrak{B}' - \mathfrak{H}' = 4\pi\mathfrak{J} \qquad [\S\ 11,\ \text{eq. } (13)]$$

By subtracting the above equation (10) from (IV) we obtain

$$\mathfrak{Z} - \mathfrak{Z}' = \frac{1}{8\pi}\mathfrak{B}'^2 - \frac{1}{4\pi}\mathfrak{B}'\mathfrak{H}' + \frac{1}{8\pi}\mathfrak{H}'^2$$

or

$$(11) \qquad \mathfrak{Z} - \mathfrak{Z}' = \frac{1}{8\pi}(\mathfrak{B}' - \mathfrak{H}')^2 = \frac{16\pi^2\mathfrak{J}^2}{8\pi} = 2\pi\mathfrak{J}^2$$

This difference $2\pi\mathfrak{J}^2$ between the values of the resultant tension in the ferromagnetic substance and in the ideal slit,

can only be due to the two terminal faces of the latter. On these, magnetic end-elements are present, the strength of which per unit surface is $+\mathfrak{J}$ or $-\mathfrak{J}$. As we have already proved (§ 21), an apparent attraction is exerted by the magnetic end-elements of one surface on those of the other, the latter being of opposite sign.

In the language of the old theory, the two faces, being charged with imaginary fluids of opposite signs and uniform density $\pm\mathfrak{J}$, attract one another, just like the two plates of a plane air-condenser charged to the electric surface-density $\pm\mathfrak{J}$. This attraction may be simply calculated,[1] and has the value, $2\pi\mathfrak{J}^2$ per unit area, which we found in equation (11).

§ 103. **Theoretical Lifting Force of a Diametrically-divided Toroid.**—Let us first consider for the sake of simplicity a toroid uniformly and closely wound, and having a uniform peripheral distribution of magnetisation, while at the same time it is cut through radially at two diametrically opposite places (fig. 26). Let us assume the faces quite smooth, and polished, so that the two halves of the toroid fit closely to each other. The width of the cut is then as small as possible, so that it differs as little as possible from the ideal cut postulated in the last paragraph. Its demagnetising action[2] as well as the leakage we may assume to be infinitely small; how far this is actually the case will be subsequently discussed.

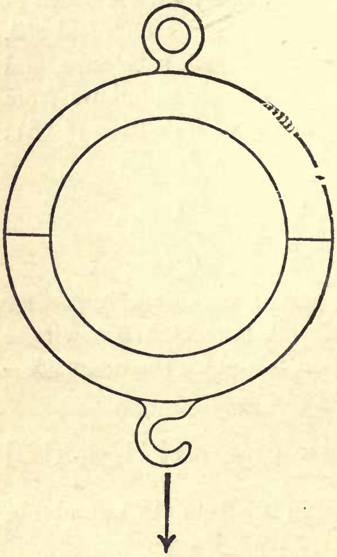

FIG. 26

Let us now investigate the attraction which the two halves exert on each other—that is, supposing we determine it by hanging weights to the lower half,

[1] See, for instance, Mascart and Joubert, *Electricity and Magnetism*, vol. 1, § 81.

[2] Compare the theory of toroids with more than one radial slit, § 81.

the greatest lifting force of the diametrically-divided toroid. The double sectional area being S, the force of attraction $\mathfrak{F}$ of both slits is

(12) . . $$\mathfrak{F} = \mathfrak{Z} S = \frac{1}{8\pi} \mathfrak{B}'^2 S$$

The value of the force $\mathfrak{F}$ is given by (12) in absolute measure, *i.e.* in dynes, provided $\mathfrak{B}'$ and S are expressed in C.G.S. units. If, on the contrary, it is to be expressed in kilogrammes-weight, and denoted by $\mathfrak{F}_1$ for distinction, we have

(12*a*) . . $$\mathfrak{F}_1 = \mathfrak{Z}_1 S = \frac{1}{8000\,\pi g} \mathfrak{B}'^2 S$$

where g is the acceleration of gravity; if we take the integral number[1] 981 cm. per second per second for this we have

$$\mathfrak{F} = \mathfrak{Z}_1 S = \frac{1}{24700000} \mathfrak{B}'^2 S$$

or with an approximation[2] sufficient for most purposes

(12*b*) . . $$\mathfrak{Z}_1 = \frac{\mathfrak{F}_1}{S} = \left(\frac{\mathfrak{B}'}{5000}\right)^2$$

Considering that under ordinary circumstances the value of induction practically attainable in a soft iron toroid is scarcely 20,000 C.G.S. units, it follows from the approximate formula (12*b*), that the corresponding tension, that is the lifting force per unit area, is about 16 kilogrammes-weight per cm^2. This

[1] Strictly speaking, the lifting force of a magnet depends on the latitude, owing to the variability of g.

[2] As the dimensions of a tension are the same as those of a hydrostatic pressure, or more generally as those of any stress (§ 101), it can also be expressed in atmospheres; one atmosphere is equal to 1·0136 megadynes per cm.^{2}; introducing this into the equation, and distinguishing the vectors by the suffix 2, we have

$$\mathfrak{Z}_2 = \frac{\mathfrak{F}_2}{S} = \frac{1}{25400000} \mathfrak{B}'^2$$

or again, with tolerable approximation,

(12*c*) . . . $$\mathfrak{Z}_2 = \frac{\mathfrak{F}_2}{S} = \left(\frac{\mathfrak{B}'}{5000}\right)^2$$

Thus from equation (12*b*) we obtain the tension in kilogrammes-weight per cm.^{2} about 1·5 per cent. too small (compare Table VI, p. 166), and from the equation(12*c*) in atmospheres too great by about the same percentage.

value is therefore practically the upper limit for the tension. The highest attainable value $\mathfrak{B} = 60{,}000$ C.G.S. would theoretically correspond to a pull of almost three hundredweight (144 kilogrammes-weight) per sq. cm. (compare § 175).

§ 104. **Resolution and Interpretation of Maxwell's Equation.** Maxwell's equation (IV) p. 157 for the resultant tension in the gap

$$\mathfrak{Z} = \frac{1}{8\pi}\mathfrak{B}'^2$$

may be resolved into three terms by once more taking into account the fundamental relation

$$\mathfrak{B}' = 4\pi\mathfrak{J} + \mathfrak{H}'$$

We have then

$$\mathfrak{Z} = \frac{1}{8\pi}(16\pi^2\mathfrak{J}^2 + 8\pi\mathfrak{J}\mathfrak{H}' + \mathfrak{H}'^2)$$

or

$$(13) \qquad . \qquad . \quad \mathfrak{Z} = 2\pi\mathfrak{J}^2 + \mathfrak{J}\mathfrak{H}' + \frac{1}{8\pi}\mathfrak{H}'^2$$

Now it may be shown that, assuming for a moment a direct action at a distance, each of the three terms of equation (13) has its own significance, which can be physically interpreted, and which in conclusion we will briefly discuss.

1. $2\pi\mathfrak{J}^2$ corresponds to the true magnetic pull between the two halves of the toroid, which can be represented by the attraction of the fictitious fluid on the faces bounding the slits (§ 102). This term, for instance, would alone come into play if the magnetisation were solely due to hysteresis of the material; that is if the value of $\mathfrak{H}^1$, being zero, $\mathfrak{J}$ had a finite value depending on magnetic retentiveness. Under ordinary circumstances this first term of equation (13) far exceeds the two others.[1]

2. $\mathfrak{J}\mathfrak{H}'$ corresponds to the electromagnetic pull of the lower half of the magnetising coil on the upper half of the ferromagnetic toroid and *vice versâ*.

[1] Stefan (*Wiener Berichte*, vol. **81**, p. 89, 1880) has given a formula for the lifting force, which contains this term only. Its approximation to the complete equation (13) is accordingly of the same order as that obtained when we write $\mathfrak{B}' = 4\pi\mathfrak{J}$ (§11, 59).

3. $\mathfrak{H}'^2 / 8\pi$ represents the purely electrodynamic action of the two halves of the magnetising coil upon each other.

If the windings are not closely coiled on the toroid, as has been assumed above, so as to form as it were a single rigid body, it may become necessary to distinguish carefully the three terms. A detailed discussion [1] of all possible conditions of experiment would lead too far. It need merely be observed that in practice the last (electrodynamic) term $1 / 8\pi \mathfrak{H}'^2$ is very small in comparison with the second (electromagnetic) one, which in turn, as we have said, is considerably exceeded by the first (magnetic) term; the right-hand side of the equation is thus arranged according to decreasing values of its three terms.

D. *Magnetic Lifting Force*

§ 105. **Older Investigations.**—The theory of the lifting force of magnets discussed in the foregoing paragraphs is as comprehensive as it is simple; it is fully expressed in Maxwell's fundamental equation (12) or (12*a*). But, however simple the theory, it is difficult to test it by experiments not open to objection, so that it can scarcely be considered as having up to now been rigorously established by experiment throughout the entire available range of induction. It seem indeed probable that the divergencies which up to the present exist are not to be ascribed to the insufficiency of the theory, but to the peculiar difficulties met with in the method of detachment which at first sight appears so easy.

A great number of investigations of all kinds have been made on the lifting force of magnets of the most varied form, electromagnets as well as permanent magnets. Only those of Dub, Lamont, Nicklès, du Moncel, dal Negro, Joule, von Waltenhofen, W. v. Siemens, &c., need be mentioned,[2] some of them having led to a series of empirical formulæ for the lifting force.

We shall here restrict ourselves to a discussion of some of the more recent investigations; in the first place, because these

[1] Compare S. P. Thompson, *Phil. Mag.* [5], vol. **26**, p. 70, 1888; du Bois, *ibid.* vol. **29**, p. 294, 1890; Shelford Bidwell, *ibid.* vol. **29**, p. 440, 1890.

[2] See the summaries in the following works: Dub, *Elektromagnetismus*, Leipzig, 1861; G. Wiedemann, *Lehre v. der Elektricität*, 3rd edition, vol. **3**, §§ 666–682 and 717–745, Brunswick, 1883; Silv. P. Thompson, *The Electromagnet*, Cantor Lectures, London, 1890.

have been made either from the point of view of the theoretical principles laid down in the preceding paragraphs or with a view of testing them; and, in the second place, because they are based on the absolute system of measurements, and this gives them far greater value than the older researches. For insuperable difficulties in interpreting or critically discussing those earlier results arise from the circumstance that in these, for some reason or other, either only relative measurements or none at all are given by their authors.

§ 106. **Wassmuth's Experiments.**—Wassmuth was the first to make accurate observations based on absolute measurements, and his experiments were arranged [1] with the express object of satisfying the conditions of theory as far as possible.

He used—

1. An iron toroid ($r_1 = 5{\cdot}84$ cm.; $r_2 = 0{\cdot}30$ cm.; $2S = 0{\cdot}565$ sq. cm.; *see* fig. 15, p. 109).

2. A welded hoop of rolled iron ($r_1 = 5{\cdot}69$ cm.; $\rho = 0{\cdot}55$ cm.; $\zeta = 1{\cdot}97$ cm.; $2S = 2{\cdot}17$ cm^2. fig. 14, p. 106).

Both were cut through diametrically, and the surfaces of contact were carefully ground and polished; it was then possible to detach both ends of the armature (that is, the lower half of the ring) 'almost simultaneously' by means of a spring balance. Both the upper and the lower half were closely wound with primary windings, the magnetisation being determined by measuring the electromotive force induced in secondary coils wound on the armature.

The following maximum values were obtained, which are comparatively small, especially when compared with those given in the following paragraphs:

With 1: $\mathfrak{H} = 93$ C.G.S.; $\mathfrak{J} = 1308$ C.G.S.; $\mathfrak{Z} = 8{\cdot}4$ kilogrammes-weight per square centimetre.

With 2: $\mathfrak{H} = 138$ C.G.S.; $\mathfrak{J} = 1157$ C.G.S.; $\mathfrak{Z} = 6{\cdot}6$ kilogrammes-weight per square centimeter.

Wassmuth found with reference to the relation between $\mathfrak{Z}$ and $\mathfrak{J}$ that the former quantity does not increase so rapidly as $\mathfrak{J}^2$, as it should do from the first term of equation (13), but that it increases more rapidly than $\mathfrak{J}$; at the same time a peculiar

[1] Wassmuth, *Wien. Ber.* vol. **85**, p. 327, 1882.

behaviour was observed in the region of maximum susceptibility. Similar results had been obtained by W. von Siemens in the investigation mentioned in § 105 on what may be regarded as very broad hoops—that is to say, pieces of tubing cut parallel to their axis. This investigation need not be further discussed here.

§ 107. **Bidwell's Experiments. Sources of Error.**—Experiments have further been made by Shelford Bidwell with a welded toroid of soft charcoal iron ($r_1 = 3{\cdot}76$ cm.; $r_2 = 0{\cdot}24$ cm.) cut through diametrically.[1] The contacts were finely polished, but had retained nevertheless a slight convexity. Each half was closely wound with nearly 1,000 turns of wire; in consequence of this the relatively strong field intensity of 585 C.G.S. units was obtained, the pull amounting to 15·9 kg.-weight per cm.^{2}, and being therefore almost equal to the value given above (§ 103) as the maximum which could in practice be obtained; such high values as these had, at that time, not been reached by any observer.

Bidwell made no measurements of magnetisation or of induction by means of secondary coils, so that his experiments are not suited for testing equation (13) experimentally. He rather assumed this as correct (restricting himself to the first two terms), and by its aid he calculated from the observed values of the pull the corresponding magnetisation as a function of the magnetising field. In § 226 we shall return to this method, which forms the foundation of certain modern methods of measurement and apparatus.

It will now be proper to consider the sources of error which always affect the apparently simple determination of the lifting force by detachment. We can no more realise in practice the theoretical assumption of an infinitely narrow slit (§ 102) than we can neglect the demagnetising and leakage-effects due to the slits; in short, their treatment as ideal cuts is inadmissible. For, however carefully the ends in contact are planed and polished, experiment shows that each slit gives rise to irregularities of that kind which can only be got rid of by using strong external pressure, and which perhaps the

[1] Shelford Bidwell, *Proc. Roy. Soc.* vol. 40, p. 486, 1886.

existence of a surface of separation necessarily entails. (Compare § 151.)

On the other hand, the grinding and polishing of the surfaces have the disadvantage that the natural adhesion may be considerable, so that the lifting force appears too great, especially with feeble magnetisation. Yet the chief source of error may perhaps be sought in the indefiniteness of the 'detachment' of such surfaces in contact, especially where two places of contact are concerned. Wassmuth, as stated above, speaks of an 'almost simultaneous' detachment; but this seems undoubtedly to mean that, in fact, one place of contact is broken first, by which an air-gap is formed which must at once demagnetise the entire toroid; owing to the decrease of induction thereby produced the other contact must also soon yield. It is questionable whether, with the otherwise favourable conditions of the toroid, less reliable results are not attained than with a bisected bar in which there is only one place of contact; an elongated ovoid divided equatorially would probably be most suitable [see § 108].

Yet even with a single contact there is uncertainty, as has always been shown since the earliest of such determinations. The more recent experiments on the action of such surfaces to be mentioned further on (§ 15, &c.), will partly explain this uncertainty, but we are not in a position to remove it. It must be assumed that a loosening of contact gradually occurs, and that in general this will first give way in one place; in this case we determine, as in all experiments on strength of materials, the resistance of the weakest part, and not the mean resistance, which is the really important matter.

Just as magnetisation gives rise to a condition of stress in the ferromagnetic substance, so also a stress produced by external forces exerts an influence on a magnetisation already existing.[1] This action is however comparatively small, so that errors arising in this manner from the feeble external forces which come into play in experiments on detachment may be neglected in comparison with the sources of error already mentioned.

§ 108. **Bosanquet's Experiments.**—Bidwell has likewise (*loc. cit.*) described some measurements on divided bars, but in this

[1] For the details of this phenomenon we refer to Ewing, *Magnetic Induction in Iron* (chapter ix.).

case also he did not determine the induction. This case has been the subject of measurements by Bosanquet.[1] The arrangement he adopted is represented in fig. 27, the cylindrical iron core consisting of two pieces each 20 cm. long by 0·526 cm. in diameter, and each wound with 1096 turns of wire. The upper electromagnet was rigidly fixed to a table, while the lower one, together with its magnetising coil, had freedom to move up and down between two brass guides (not shown in the figure) with as little friction as possible. The ends of the two pieces were ground against each other so as to ensure perfect contact. To the lower electromagnet was attached a scale-pan; the weight was counterpoised, as shown in the figure, so that the weights placed in the pan gave a direct measure of the lifting force. Near the division was a small secondary coil, so that the induction could be measured by reversing the primary current. Weights were added until the lower electromagnet was detached, and struck against a stop, at a few millimetres distance.

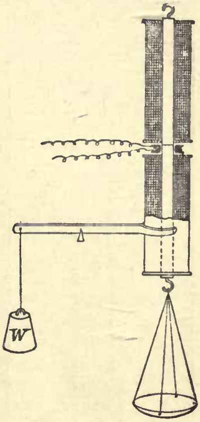

FIG. 27

The highest value obtained for the pull was 14·6 kg.-weight per square cm. with an induction of 18,500 C.G.S. units. On the whole, the equation

$$\mathfrak{F}_1 = \frac{1}{8000\,\pi g}\,\mathfrak{B}'^2 S$$

may be regarded as roughly confirmed by Bosanquet's experiments. The deviations may be attributed to the various recognised sources of error.[2]

It follows from all this that Maxwell's fundamental law of magnetic tension as expressed by equation (12) § 103 is

[1] Bosanquet, *Phil. Mag.* [5], vol. 22, p. 535, 1886.

[2] Besides the irregular deviations, which are to be referred to the uncertainty of the detachment already discussed, Bosanquet constantly found too high lifting forces, especially for small values of the induction. This might be due to friction against the brass collars besides the ordinary adhesion of the surfaces in contact. The mutual electrodynamic action of the two coils also was greater than $\mathfrak{H}'^2 S/8\pi$ (corresponding to the last term of equation

to be considered as experimentally confirmed with tolerably close approximation; experiments perfectly free from objection would, however, be very desirable.[1]

We give finally in Table VI a series of corresponding values of the induction $\mathfrak{B}'$ and of the tension $\mathfrak{Z}$; in the fourth column the values[2] are given according to the rigorous equation (12*a*), in the third according to the convenient approximate equation (12*b*); the two sets of values agree closely, as will be seen. (Compare the remark p. 159.)

TABLE VI

$\mathfrak{B}'$ C. G. S. unit	$\frac{\mathfrak{B}'}{5000}$	$\mathfrak{Z}$ Kg.-weight per sq. cm.	
		Eq. (12*b*)	Eq. (12*a*)
1000	0·2	0·04	0·041
2000	0·4	0·16	0·162
3000	0·6	0·36	0·365
4000	0·8	0·64	0·649
5000	1·0	1·00	1·014
6000	1·2	1·44	1·460
7000	1·4	1·96	1·987
8000	1·6	2·56	2·596
9000	1·8	3·24	3·286
10000	2·0	4·00	4·056
11000	2·2	4·84	4·907
12000	2·4	5·76	5·841
13000	2·6	6·76	6·855
14000	2·8	7·84	7·550
15000	3·0	9·00	9·124
16000	3·2	10·24	10·39
17000	3·4	11·56	11·72
18000	3·6	12·96	13·14
19000	3·8	14·44	14·63
20000	4·0	16·00	16·23

(13)), for, owing to their considerable diameter, their sectional area may be far greater than that of the bar.

[1] In connection with this it may be mentioned that the corresponding Maxwellian equations for paramagnetic and diamagnetic substances of constant susceptibility, which, for our present purpose, we regard as magnetically indifferent, have been completely confirmed by experiment; so far, at any rate, as the experiments of Quincke and others with the 'U-tube' methods extend. See du Bois, *Wied. Ann.* vol. 35, p. 137, 1888, where the literature on the subject is collated. (Compare also § 203.)

[2] Taken from the Cantor Lectures on the Electromagnet by Professor Silv. Thompson.

[Since the above was originally published, experiments by Threlfall appeared in the *Phil. Mag.* for July 1894, made by a method essentially like that of

§ 109. **Conclusions from Maxwell's Law.**—Maxwell's law constitutes the common foundation on which are based all considerations as to magnetic tension, attraction, lifting force, &c.[1] Maxwell, who only mentions the law casually in a few sentences,[2] has thus laid the foundation for a rational understanding of this question, while the great number of experimental investigations previously made were unable even remotely to suggest an explanation. We shall discuss in conclusion some aspects of the question, which are to be kept in view, in applying the law to the various kinds of magnetic circuits occurring in practice.

At each point of the surface of an ideal slit [equation (IV) § 102] the magnetic tension—that is, the pull per unit surface of the gap—is

$$\mathfrak{Z} = \frac{1}{8\pi}\mathfrak{B}'^2$$

If now the induction $\mathfrak{B}'$ has the same value over the whole area S of the surface, as was assumed in the theoretical treatment of the toroid (§ 103), the flux of induction through it is

$$\mathfrak{G}' = \mathfrak{B}' S$$

hence the whole force $\mathfrak{F}$ due to this tension will be

$$(14) \quad . \quad . \quad \mathfrak{F} = \frac{1}{8\pi}\mathfrak{B}'^2 S = \frac{1}{8\pi}\frac{\mathfrak{G}'^2}{S}$$

This equation may be expressed in words as follows:

When the induction is given, the tension is directly proportional to the area; when the flux of induction is given it is inversely proportional to the area.

The latter principle, which at first appears somewhat strange,

Bosanquet. Mr. Taylor Jones, acting on the suggestions contained in the text, made a complete and careful investigation on the validity of Maxwell's law by the divided-ovoid method. He found it confirmed for inductions up to 19,000 C.G.S. units, so that his experiments appear to settle the question so far, it being left to the investigation of a higher range of induction to produce further evidence (*Phil. Mag.*, March 1895). H. du Bois.]

[1] In many special cases, Coulomb's law is as before conveniently made use of, § 21. This, however, is closely related to that of Maxwell.

[2] Maxwell, *Treatise*, 2nd edit. vol. **2**, § 642.

is a simple deduction from the quadratic law of tension. To understand this more clearly, let us imagine a magnetic circuit, and first of all without any leakage, so that the flux of induction has the same value in every part of it. The total tractive force on a section will then be greater the smaller the section, and will in fact be inversely proportional to it. The unavoidable leakage entails, however, an important limitation, so that actually, as the cross-section is diminished, the tractive force will attain a maximum value, and will then again decrease.

It is here expressly assumed that the flux of induction is in some way or other kept constant, so that when the section is diminished at one or more parts of the magnetic circuit, a greater mean magnetic intensity is required. For with given values of the magnetising current, and of the number of windings, every such 'throttling' of the magnetic circuit will directly produce a diminution of the mean flux of induction; that follows directly from the Hopkinson theory, or from the considerations in the following chapter.

§ 110. **Load-Ratio of a Magnet.**—There are numerous experimental confirmations, or simple examples of the conclusions in the foregoing paragraphs, for which we may refer to Chapter IX. as well as to the work of Professor Silvanus Thompson already quoted. A further deduction from the fundamental law

$$\mathfrak{F} = \frac{1}{8\pi}\mathfrak{B}'^2 S$$

is this, that the question so frequently occurring in the older literature as to the ratio of the lifting force to the weight of a magnet or electromagnet is quite unimportant. For with similar electromagnetic systems, with currents proportional to the linear dimensions (§ 67), the values of the magnetic intensity, and consequently of the induction, will be the same in corresponding places. Hence the lifting force of corresponding surfaces of contact in the two systems, will be proportional to their section—that is, to the square of their linear dimensions. On the other hand, the weight of the magnet itself is proportional to the third power of its linear dimensions; and it follows from this that the *load-ratio* (lifting-force/weight) in similar magnets will be inversely proportional to the linear

dimensions. Thus large magnets have in this respect less favourable dimensions, while the relative load-ratio may theoretically be infinitely increased by making the magnet small enough.

This is in accordance with experiment. Silv. Thompson mentions[1] a small electromagnet weighing 0·1 gramme which could lift 250 grammes—that is, 2500 times its own weight.

If the induction over the area of contact is not constant but variable, the resultant force $\mathfrak{F}$ is manifestly given by the surface integral

$$(15) \qquad . \quad . \quad . \qquad \mathfrak{F} = \frac{1}{8\pi} \iint \mathfrak{B}'^2 \, dS$$

taken over the section S. This value, by a well-known theorem, will be greater than

$$\mathfrak{F}' = \frac{1}{8\pi} \overline{\mathfrak{B}}'^2 S = \frac{1}{8\pi} \frac{\mathfrak{G}'^2}{S}$$

which would be obtained by introducing the arithmetical mean value of the induction

$$\overline{\mathfrak{B}} = \frac{1}{S} \iint \mathfrak{B}' \, dS = \frac{\mathfrak{G}'}{S}$$

An unequal distribution of the induction is therefore in a certain sense advantageous, provided care is taken to ensure the constancy of the value of its surface-integral over the cross-section of the magnetic circuit—the constancy, that is, of the flux of induction.

[1] Silv. Thompson, *loc. cit.* p. 34. It is there shown how the relation deduced above may be so expressed that the lifting force is proportional to the two-third power or the $\frac{3}{2}$ root of the magnet's weight which agrees with the older Bernouilli-Häcker empirical formula (see also *Phil. Mag.* [5], vol. **26**, p. 70, 1888).

CHAPTER VII

ANALOGY OF THE MAGNETIC CIRCUIT WITH OTHER CIRCUITS

A. *Historical Survey*

§ 111. **Older Development. First Stage.**—The idea of an analogy of magnetic systems with systems of circuits of various kinds (hydrokinetic, thermal, galvanic) is already, in 1761, met with in Euler.[1] In contradistinction to Poisson's hypothesis of two fluids (§ 27), which at that time had not yet been elaborated, he assumed a single subtile kind of matter, which was supposed to flow with great velocity through the magnets, as well as through the surrounding air-space. The ferromagnetic substance offering a far less resistance than the indifferent surrounding medium, it tends preferably to pass through that substance.

The paths of that subtile matter are assumed by Euler to be identical with the lines mapped out by iron filings, and he therefore lays considerable weight on the determination of these figures. From the tendency of the magnetic matter to take its course, as far as possible, through the ferromagnetic substance, which is supposed, for this purpose, to possess innumerable fine channels, he explains a number of phenomena in a manner which surprisingly suggests our present mode of expression. Those very elementary letters and figures contain a remarkable forecast of views which were only to be completely developed more than a century later.

In the later literature we repeatedly find indications—very obscurely conceived, it is true—of magnetism being something flowing in closed paths, combined with the assumption

[1] Euler, *Briefe an eine deutsche Prinzessin*, vol. 3, pp. 95–150, Leipzig, 1780. These considerations apply exclusively to permanent magnets, since the connection between electricity and magnetism was at that time unknown.

that this flow takes place more easily in iron than in indifferent media. Cumming,[1] so long ago as 1821, made researches on what, in this sense, was called magnetic conductivity.

In the various writings, moreover, of Ritchie, Sturgeon, Dove, Dub, and de la Rive the theory of such closed magnetic circuits is more or less clearly discussed. Joule, who, in the first years of his scientific activity, occupied himself greatly with the investigation of electromagnetic machines, enunciates in one place [2] the following principle:—

The maximum power of an electromagnet is directly proportional to its least transverse sectional area.

On the other hand (*loc. cit.* p. 36), he maintains that the 'resistance' to induction varies in the direct ratio of the *length* of the (closed) electromagnet. If we compare these and some other statements in Joule's writings, we obtain an approximately correct representation of the modern views on this subject, which only wants the mathematical clothing.

§ 112. **Continuation (Faraday, Maxwell).**—The question entered into a more advanced phase by the conceptions introduced by Faraday, to illustrate which, he was fond of using lines of force (§ 65).[3] His theoretical views were, indeed, neither understood nor appreciated by most of his contemporaries, although his experimental inquiries evoked the highest admiration. It cannot, indeed, be denied that much want of clearness attached to Faraday's views, which was only afterwards gradually removed, mainly by the exertions of Maxwell.

Faraday showed, in the first place, that his lines of force must always form closed curves, the course of which is influenced by the magnetic 'conductivity' of the medium traversed. He, also, was the first to compare an electromagnet with a voltaic pile, and, in order to complete the analogy, he supposed it immersed in an electrolyte, the finite conductivity of which was the analogue of the finite permeability, equal to unity, of the air surrounding the ferromagnetic substance; for in the ordinary arrangement of electrical circuits the conductivity of the surrounding medium is obviously zero, or, at any rate, excessively

[1] Cumming, *Cambridge Phil. Soc. Trans.* 1821.

[2] Joule, *Reprint of Scientific Papers*, vol. 1, p. 34, London, 1884.

[3] Faraday, *Exp. Res.* vol. 3, pp. 328–443.

small. The analogy in question was frequently referred to by later authors (§§ 123, 133).

To Maxwell is due the merit of having elucidated Faraday's views, and of clothing them in a mathematical form. Instead of lines of force, he introduces the conception of tubes of induction (§ 63). In one part of his work he expresses himself as follows:[1]—

'The problem of induced magnetism, when considered with respect to the relation between magnetic induction and magnetic force, corresponds exactly with the problem of the conduction of electric currents through heterogeneous media.

'The magnetic force is derived from the magnetic potential, precisely as the electric force is derived from the electric potential. The magnetic induction is a quantity of the nature of a flux, and satisfies the same conditions of continuity as the electric current does.

'In isotropic media the magnetic induction depends on the magnetic force in a manner which exactly corresponds with that in which the electric current depends on the electromotive force. The magnetic inductive capacity in the one problem corresponds to the specific conductivity in the other.'[2]

These principles form the kernel of the later development of the subject.

§ 113. **Continuation (Lord Kelvin).**—In his theory of magnetism Lord Kelvin has repeatedly[3] dwelt on the complete analogy which exists between the mathematical theories of magnetic induction, of dielectric polarisation, and of Fourier's theory of thermal conduction, on the one hand, and the theory of certain hydrokinetic processes on the other.

He has further shown, referring to the above-mentioned speculations of Euler, that the vector which we have called the magnetic intensity $\mathfrak{H}$ cannot be likened to the velocity of an incompressible liquid, for this leads logically to the assumption of a creation or destruction of that liquid in the places where the

[1] Maxwell, *Treatise*, 2nd edition, vol. **2**, p. 51.

[2] *Ibid.*; the last passage obviously refers to the assumption of a constant permeability; its significance, as we shall presently see, is essentially limited by the fact that this assumption is not in accordance with the facts.

[3] Sir W. Thomson, *Reprint of Papers on Electrostatics and Magnetism*, Arts. 27, 31, 32, 41, 42.

magnetic end-elements would exist in the corresponding electromagnetic system. Lord Kelvin showed at that time (1872), on the contrary, that it is the induction $\mathfrak{B}$[1] which must be taken in this case as the analogue of the velocity of an incompressible liquid, and he enunciates the following principle, among others (*loc. cit.* § 576):

The resultant force defined electromagnetically for the space occupied by the magnet, and the resultant magnetic force according to the unambiguous definition for space not occupied by the magnet, agree everywhere in magnitude and direction with the velocity in a possible case of motion of an incompressible liquid filling all space.

In order to make the analogy more complete Lord Kelvin assumed a special kind of medium (*loc. cit.* art. 42); that is to say, a porous solid of infinitely fine-grained texture, through which filters an incompressible frictionless liquid; its motion must be irrotational, or in other words it must possess a velocity potential.[2] The more permeable is such a porous medium, the greater will be the flow[3] of the liquid in comparison with the kinetic energy of unit volume of the space occupied by the body in question and the liquid. The ratio of these two quantities affords therefore a measure of the permeability of the porous body; it may be called the *hydrokinetic permeability*.

Having regard to the existing analogies, Lord Kelvin proposed to extend the conception of permeability and to denote

[1] The vector $\mathfrak{H}$ was called by Lord Kelvin (*loc. cit.* § 479) the 'resultant magnetic force,' provided we are dealing with some point in the magnetically indifferent region. At points within the ferromagnetic substance, he further distinguishes between the 'polar definition' ($\mathfrak{H}$) and the 'electromagnetic definition' ($\mathfrak{B}$) of magnetic force; according as the cylindrical cavity (Chap. III. § 51), with its axis in the direction of magnetisation, has a diameter very small compared with its length, or a length very small compared with its diameter. He expressly opposes (*Math. and Phys. Papers*, vol. **3**, p. 478) the name 'induction' given to $\mathfrak{B}$ by Maxwell. (Compare Chap. I. p. 13, footnote.)

[2] If the components of velocity satisfy the hydrodynamical equations of irrotationality (eq. (7), § 39), the distribution of the velocity, from what has been said, is a lamellar one, and this vector has therefore a 'velocity potential.'

[3] By flow is understood the quantity of liquid which per unit time passes through unit section of the space occupied by the porous body and the liquid.

by the same expression the analogous property in the four analogous theories adduced by him; in addition to the hydrokinetic permeability already defined, he therefore distinguished (*loc. cit.* art. 31).

1. *Magnetic Permeability*; ratio of magnetic induction to magnetic intensity.

2. *Dielectric Permeability*; identical with dielectric constant, or [specific inductive capacity].

3. *Thermal Permeability*; identical with thermal conductivity.

These analogies are then illustrated by special examples, with calculations and graphic representations. The discussions in question comprise (*loc. cit.*) more than fifty articles, of which we can here only give a short sketch. Having regard to the modern developments and the wide diffusion of the ideas which form its basis, those researches are at the present time of the greatest interest.[1]

§ 114. **Summary.**—It may from all this be reasonably maintained, that the mathematical analogy of the theory of electromagnetic systems with other well-known physical theories, has not only been suggested by the publications of Euler, Faraday, Maxwell and Lord Kelvin, but even worked out in all particulars, especially by the last mentioned, and in an unassailable manner. Correct interpretation only was needed to apply these mathematical results to practical problems.

The introduction of these views into the domain of applied science, to which we shall now turn our attention, has only been effected within the last decade. If we consider that many errors have thus crept in which might lead to obvious inconsistency, we can understand why this modern development has hitherto been little considered by some physicists, being regarded as not even new, and at the same time partly incorrect.

We shall proceed most safely if we maintain from the outset that that modern development plays, and will always

[1] It may finally be observed that besides the purely mathematical hydrokinetic analogy a more profound hydrodynamical one is discussed (*loc. cit.* art. 41); in connection therewith the experiments of Schellbach and of Guthrie on repulsion and attraction due to vibrations in a liquid are discussed; this subject, as is well known, has recently been subjected by Bjerknes to detailed mathematical and experimental treatment.

play, only a subordinate part from the purely scientific point of view. Its undoubtedly great success in practical applications compels us, however, to deal with it more in detail; we shall, therefore, endeavour at the outset to anticipate such errors as are most likely to arise, and thereby prevent our attaining incorrect results. We will, however, first extend the historical notices over the last decade, giving them in the chronological order of their publication.

§ 115. **Recent Development (Rowland).**—In September 1884 Rowland proposed a formula for the number of lines of force in the field magnets of a dynamo.[1] He formed a fraction the numerator of which was the product of the current into the number of turns (what is now called the ampère-turns); the denominator was a complicated expression which was to represent the 'magnetic resistance' of the iron and of the air; the leakage being taken into account. We have spoken of this proposal of Rowland, in the first place, because mention of it is to be found in an extended experimental investigation, published as long ago as 1873.[1] He then wrote [*loc. cit.*, p. 145, equations (3) and (4)] the fraction in question as follows:

$$(1) \quad . \quad . \quad . \quad . \quad Q' = \frac{M}{R}$$

which in our notation, to be afterwards described, would be expressed by the formula

$$(2) \quad . \quad . \quad . \quad . \quad \mathfrak{G} = \frac{\mathfrak{H}}{\left(\frac{X}{L}\right)}$$

Rowland's R, our $\frac{X}{L}$, is defined by him as the magnetic resistance per unit length. Hence in the above formulæ numerator and denominator have only to be multiplied by L, the length of the portion of magnetic circuit considered to bring them to the form which is now most usual. [§ 119, equation (I).]

The principle, so important in practice, that the magnetic

[1] H. A Rowland, *Electrician*, vol. **13**, p. 536, 1884.

[2] Ibid., *Phil. Mag.* [4], vol **46**, p. 140, 1873.

resistance is to be made as small as possible was applied in 1879 in the Elphinstone-Vincent six-polar dynamo machine,[1] and was expressly discussed by the inventors; but that this machine, owing to practical difficulties in the construction, has found no general reception, by no means diminishes its historical interest.

§ 116. **Continuation (Bosanquet).**—In March 1883 Bosanquet developed the mathematical analogy of magnetism with electrical conduction,[2] and made experiments in this direction with closed rings. He first used the expression 'magnetomotive force' as an analogue of electromotive force. He defined the former quantity simply as a difference of magnetic potential, just as in many cases electromotive force is only another expression for difference of electrical potential. Instead of taking the magnetic intensity $\mathfrak{H}$ as the starting-point, he took its line-integral—that is, the magnetomotive force M just defined. Against the adaptation of this conception, which had long before been introduced, nothing was to be said. The name alone was new, and we cannot consider it as a happy choice. The same objection may of course be as well raised against its prototype, 'electromotive force,' which is, however, so generally adopted that there can be no question of changing it.

Bosanquet further retained the induction $\mathfrak{B}$, and called $M/\mathfrak{B}$ the magnetic resistance, although it seems from his paper as if he saw that the cross-section should also have been taken into account. In fact, in the analogy carried out by him, the flux of induction $\mathfrak{G}$, and not the induction $\mathfrak{B}$, corresponds to the electrical current. If, then, a magnetic resistance is at all to be introduced, we must choose the ratio $M/\mathfrak{G}$, as has since then been done by all authors, and not $M/\mathfrak{B}$.

§ 117. **Continuation (W. von Siemens).**—In 1884 Werner von Siemens, in his 'Contributions to the Theory of Magnetism,'[3] occupied himself with this subject, experimenting with closed

[1] See Silv. Thompson, *Dynamo-Electric Machinery*, 3rd edition, p. 211; 4th edition, p. 172. A comparison of the different editions of this book gives a true picture of the historic development of this subject.

[2] Bosanquet, *Phil. Mag.* [5], vol. **15**, p. 205, 1883; and vol. **19**, p. 73, 1885.

[3] W. von Siemens, *Berl. Sitzungsberichte*, Oct. 1884; *Wied. Ann.* vol. **24**, p. 93, 1885.

and open electromagnetic circuits. He summarises his results as follows in the appendix to his 'Recollections':—

'We may thus, after Faraday, also conceive the magnetic action at a distance as proceeding from molecule to molecule, or from volume-element to volume-element, and are then justified in applying to magnetism the laws of molecular transfer which hold for heat, current-electricity, and electrostatic distributions.

'This theory requires, again, the assumption that magnetism, like the electric current and electric distributions, can only exist in closed circuits in which the magnetic moment is inversely proportional to the resistance of the circuit. This view leads, consequently, to the introduction of the ideas of "magnetic distributive resistance" and "magnetic conductivity" of space and of magnetic bodies. Hence, only as much magnetism can be developed in an iron bar by an electrical current circulating round it as can pass or be conducted from one end of the bar to the other through the space round the magnet. My experiments have confirmed this view, and they have shown that the magnetic conductivity of soft iron is, approximately, five hundred times as great as that of non-magnetic substances or of vacuum.

'Hence, Ohm's law may be applied in the construction of electromagnetic machines to ascertain the most suitable dimensions which may be of advantage to the electrician. The idea of magnetic conductivity, which, so far as I know, I first introduced, has meanwhile been frequently used and further developed in technical works, without, however, any mention of my name.'

As regards the latter passage, it must be left to the reader to form a judgment as to the relative merits of the investigators concerned. The scientific epoch in question is too near for an historical and critical treatment, so that we must here restrict ourselves to giving as impartial a review as we can of the literature in question.

In order further to elucidate the above quotation, which is expressed in popular language, we add the following explanation, and refer to the original paper for the rest.

The following is enunciated as a general law (*loc. cit.* p. 95):—

The 'Strength of Magnetism' [b] is equal to the 'Sum of the magnetising forces' [e], divided by the 'Sum of the opposing Resistances' [i]. (Compare §§ 119 and 124.)

In our notation we must read instead of [b] : flux of induction : instead of [e] line-integral of magnetic intensity; [i] is put proportional (*loc. cit.* p. 98) to the length, and inversely proportional to the section and the magnetic conductivity of the iron. The variability of the latter quantity is discussed, and its highest value is found by experiment to be equal to about 500, the conductivity of air being assumed as the unit.

§ 118. **Continuation (Kapp, Pisati).**—Starting from the above considerations, Gisbert Kapp in 1885 gave empirical rules for the construction of dynamo machines, and these have been much used in practice with great success.[1] Kapp's rules have rather less claim to scientific value, partly because the inch and minute are used as fundamental units instead of the usual C.G.S. units.

In this mixed system of measurement the reluctivity of air is expressed by the number 1440; that of iron was first considered constant; for convenience the number 2 was taken for wrought iron, and 3 for cast iron. This rule used to give a value for the number of lines of force which, under certain circumstances, was as much as 40 per cent. in excess of that actually observed; this, no doubt, arose partly from leakage. Kapp then introduced a $\tan^{-1}$ formula for magnetisation, like those which had been given in 1850 by J. Müller and in 1865 by von Waltenhofen.[2]

More recently Pisati[3] has again brought forward the analogy between the theory of the magnetic circuit and Fourier's theory of the conduction of heat, and has experimentally investigated it. From what has been said above, this is only a special case of the general analogy. Induction must then logically, as we shall afterwards see, correspond to the rate of flow of heat (that is, to the quantity of heat passing per unit cross-section per unit time), magnetic intensity, on the other hand, to the temperature-

[1] Gisbert Kapp, *Electrician*, vols. **14**, **15**, **16**, 1885.

[2] J. Müller, *Pogg. Ann.* vol. **79**, p. 337, 1850, and vol. **82**, p. 181, 1851; von Waltenhofen, *Wiener Berichte*, vol. **52**, p. 87, 1865.

[3] Pisati, *Rend. R. Acc. Lincei*, vol. **6**, pp. 82, 168, 487, 1890.

gradient. The ratio of the two quantities is, in one case, the magnetic permeability; in the other, Fourier's coefficient of thermal conductivity. The latter is, of course, not constant, as Fourier originally supposed, but depends on the temperature. This analogy then is less faulty than that with Ohm's law, in which the electrical conductivity is absolutely constant (§ 120). In order, however, that the analogy may be mathematically correct, thermal conductivity ought not to be a function of the temperature, but of the temperature-gradient, which, so far as is known, is not the case.

In Fourier's theory a second coefficient is introduced, which allows for the loss of heat from the surface by convection, radiation, and conduction. This phenomenon, according to Pisati, is the analogue of magnetic leakage.

The case of bars and rings with local magnetising coils he compares to Fourier's well-known problem of a conducting body in which there is a given distribution of heat, and which is embedded in a cooling medium. The experiments of Pisati show, in part, a good agreement with his theory.

Still more recently papers by Steinmetz, Kennelly, Corsepius, R. Lang and O. Frölich have been added to the literature of the subject.[1] After this historical review we turn to the exposition of the present state of the views on this subject.

B. *Modern Conception of the Magnetic Circuit*

§ 119. **Definitions.**—We shall first give the definitions of the new conceptions, and illustrate these as usual by the typical example of a closed toroid uniformly magnetised. Let its mean perimeter be L as before, and the area of its cross-section be S. The whole of the quantities to be defined may be derived in the simplest manner from the vectors $\mathfrak{B}$ and $\mathfrak{H}$, and the geometrical quantities L and S.

1. *Flux of Induction* $\mathfrak{G}$ has been already defined (§ 61); it is

$$(3) \quad . \quad . \quad . \quad . \quad \mathfrak{G} = \mathfrak{B} S$$

[1] Steinmetz, *Elektrotechn. Zeitschrift*, vol. **12**, pp. 1, 13, 573, 1891; vol. **13**, pp. 203, 365, 1892; Kennelly, *ibid.* vol. **13**, p. 205, 1892; Corsepius, *ibid.* vol. **13**, pp. 243, 414, 1892; R. Lang, *ibid.* vol. **13**, pp. 734, 485, 495, 510, 522, 1892; O. Frölich, *ibid.* vol. **14**, pp. 365, 387, 401, 1893.

2. The *line-integral of magnetic intensity* along the circumference of the toroid is $(\mathfrak{H}\, L)$; this quantity is after Bosanquet pretty generally called *magnetomotive force*; we shall denote it by M. In the present case it cannot directly be considered as a difference of magnetic potential, because the magnetic potential within the space occupied by the toroid is a many-valued function. We have thus

$$M = \mathfrak{H}\, L \tag{4}$$

3. *Permeability* μ has already been defined (§ 14); it is

$$\mu = \frac{\mathfrak{B}}{\mathfrak{H}} \tag{5}$$

4. *Reluctivity* ξ has also already been introduced; it is

$$\xi = \frac{1}{\mu} = \frac{\mathfrak{H}}{\mathfrak{B}} \tag{6}$$

We shall now have to consider two other quantities which are derived from the last two by purely geometrical processes. These latter numbers only characterise the magnetic properties of the substances in question, while in the scalar quantities now to be introduced, the length and transverse dimensions of the body to be magnetised have also to be taken into account.

5. *Permeance* V is defined by the equation

$$V = \frac{\mu\, S}{L} \tag{7}$$

and its reciprocal.

6. *Reluctance* X by the equation

$$X = \frac{\xi\, L}{S} = \frac{1}{V} \tag{8}$$

Starting now from equation (5), which we write in the following form,

$$\mathfrak{B} = \mu\, \mathfrak{H}$$

and multiplying both sides by S/L, we obtain after a simple transformation

$$(\mathfrak{B}\, S) = \frac{\mu\, S}{L}\, (\mathfrak{H}\, L)$$

If we now introduce the quantities defined above we have

$$\text{(I)} \qquad . \quad . \quad . \qquad \mathfrak{G} = MV = \frac{M}{X}$$

This equation means :

I. *The flux of induction is equal to the magnetomotive force multiplied by the permeance, or, what is the same thing, divided by the reluctance.*

§ 120. **Ohm's Law.**—This proposition, in the form given, resembles Ohm's law, but is in fact different in the cardinal point, as will afterwards be demonstrated. Hence to apply Ohm's law to electromagnetic circuits without further enquiry, as has frequently been done, must be stigmatised as an unscientific proceeding. Such an application no doubt offers certain practical advantages, by allowing us to retain some familiar ideas which are associated with Ohm's law; this, however, is of course no sufficient reason for their unconditional acceptance, which cannot be admitted, more especially for the following reasons.

The quintessence of Ohm's law is the constancy of electrical resistance—that is, its perfect independence of the strength of the current. That quantity depends, as is well known, only on the nature of the conductor, its temperature,[1] and its geometrical dimensions. So far as our present experimental knowledge goes, it is also independent of the surrounding medium.[2] Maxwell, after stating Ohm's law, expresses himself as follows :—

'Here a new term is introduced, the Resistance of a conductor, which is defined to be the ratio of the electromotive force to the strength of the current which it produces. The introduction of

[1] That the temperature may in certain circumstances be raised by the heat developed by the current, and thereby the resistance may be indirectly dependent on the current, is quite unimportant, and can be prevented by the suitable application of ice-baths, &c. The temperature is not completely determined by the current. It is not, in fact, a function of the current at all, but is to be considered as an independent variable. This obvious remark is inserted here, since in some quarters opposite conclusions have been based on the heat developed by the current.

[2] Sanford's recent experiments (*Phil. Mag.* [5], vol. **35**, p. 65, 1893) are here disregarded, as they are open to a series of critical objections, and in any case appear to require confirmation.

this term would have been of no scientific value, unless Ohm had shown, as he did experimentally, that it corresponds to a real physical quantity—that is, that it has a definite value which is altered only when the nature of the conductor is altered.'[1]

Maxwell's warning that the introduction of new ideas, or of modern expressions for old ideas, had better be omitted so long as their scientific value is not demonstrated, seems not in recent times to have been sufficiently laid to heart.

After the experiments mentioned by Maxwell as having been made by Ohm himself, countless others have been tried to test this law. We need mention only those of Fechner, Pouillet, Beetz, R. Kohlrausch, and Chrystal.[2] The latter states, as follows, the results of his experiments, in which a method proposed by Maxwell was used: 'Within the very extended range of current-strength employed, the electrical resistance of a conductor does not vary by more than the billionth part of its value.' Ohm's law is therefore confirmed with an accuracy which has scarcely its parallel in physics.

§ 121. **The Magnetic Reluctance-Function.**—Let us now state Ohm's law in its usual simple form:

$$I = \frac{E}{R} \tag{9}$$

and compare with it the above magnetic equation (I)

$$\mathfrak{G} = \frac{M}{X}$$

Here, apart from hysteresis, $\mathfrak{G}$ is a function of the magnetomotive force M alone. The two quantities are, however, not proportional, since their ratio X is by no means constant, and cannot, therefore, be regarded as a resistance in Ohm's sense. For if the resistance were no longer to be regarded as constant, we might logically divide any function, however complicated, into its independent variable, and consider the quotient as a mathematical resistance to the increase of the function. We should then be in a position to represent almost every process occurring in nature by an equation corresponding to Ohm's law.

[1] Maxwell, *Treatise*, 2nd edition, vol. **1**, § 241.

[2] Wiedemann, *Lehre von der Elektricität*. 3rd edition, vol. **1**, §§ 329–353.

As regards the actual form of the magnetic reluctance-function X for a given ferromagnetic body, we may recall the definition given above [equation (8)]

$$X = \frac{\xi L}{S}$$

Since L and S are geometrical constants, it is sufficient to consider the law of the reluctivity ξ (the reciprocal of the permeability μ), as was already done in the first chapter. We refer, therefore, to the curve $\xi =$ funct. ($\mathfrak{H}$) (fig. 4, p. 21), and to the discussion of its shape. Any given curve may, as is well known, be represented approximately, within a short range, by a straight line; and the reluctivity-curve, after it has passed its minimum-point, is very nearly straight. Hence, for cast iron, for the range of abscissæ between 25 and 500 C.G.S., within which limits the field used in practice will mostly lie, the curve can be represented by the equation

$$(10) \quad . \quad . \quad . \qquad \xi = a + b\mathfrak{H} \qquad [25 < \mathfrak{H} < 500]$$

where a and b are two constants. In the present case, a will only have a small value, as the straight line in question almost passes through the origin. The latter would be exactly the case if the induction $\mathfrak{B}$ were constant within the region in question; for, according to the definition, $\xi = \mathfrak{H}/\mathfrak{B}$, and therefore, when the denominator is constant, is proportional to the numerator. Its graph is in that case a straight line through the origin. The induction within a certain interval varies in fact but little, yet it is nowhere perfectly constant, and does not tend towards any constant value.

Especial prominence has recently been given by several writers[1] to the empirical linear equation (10). It is in reality of subordinate importance, from a scientific point of view. If it is introduced into the equation

$$\mathfrak{B} = \frac{\mathfrak{H}}{\xi}$$

we obtain

$$(11) \quad . \qquad \mathfrak{B} = \frac{\mathfrak{H}}{a + b\mathfrak{H}} = \frac{1}{b} \cdot \frac{\left(\frac{b}{a}\right)\mathfrak{H}}{1 + \left(\frac{b}{a}\right)\mathfrak{H}}$$

[1] See Kennelly, *Elektrotech. Zeitschrift*, vol. 13, p. 205, 1892.

which equation, except for a constant factor $1/b$, agrees with the older formula of Frölich, to be presently discussed (§ 138). The above relation, which is purely empirical, and is to be restricted to the same range as equation (10), obviously holds only for closed rings, for the curve $\xi =$ funct. ($\mathfrak{H}$) (fig. 4, p. 21), is deduced from the normal curve of magnetisation (fig. 2, p. 18), in which self-demagnetisation is not allowed for.

§ 122. **Summary.**—We have not mentioned, or but briefly, the empirical methods which still hold their sway in magnetism; this is the more justifiable as we are now in possession of rational bases for attacking the solution of most practical questions.

The theory of Drs. J. and E. Hopkinson, treated in Chapter VI., §§ 96–100, has in this respect great importance. It unites precision with brevity and elegance; no essential objection can be raised against it. In the memoir of Drs. Hopkinson mentioned above the expressions 'magnetomotive force' and 'resistance' will be sought for in vain, and still more so any reference to a law corresponding to that of Ohm. The idea denoted by the former expression, the line-integral of intensity, is, however, introduced as an independent variable, along with the flux of induction. In consequence of this, the Hopkinsons' equation may at once be transformed into a series of terms, each of the form of equation (I), § 119, as we shall afterwards show (§ 131).

Moreover, it must again be urged that it is quite permissible, though it has no scientific value, to divide the magnetomotive force by the flux of induction, and to call the quotient the magnetic reluctance—or, its reciprocal the magnetic permeance. The perspicuity which is attained by introducing these two quantities cannot be denied, and in the following chapters we shall repeatedly have occasion to be convinced of this. The error first comes in when the magnetic reluctance thus defined, which depends, in the most arbitrary manner, on the magnetising field (or on the induction) is now compared with electrical resistance, the essential characteristic of which is its absolute constancy.

§ 123. **Leakage. Magnetic Shunts.**—The above remarks are not vitiated by the fact that in some cases approximately correct results may be obtained by applying an imitation of Ohm's

law to magnetic circuits. This may be the case especially when the permeability of the ferromagnetic substance is very great in comparison with that of the surrounding indifferent space (which is taken as unity). It will in many cases be indifferent whether the permeability has the value 200 or 2000; both numbers are so large that the practical results will not be appreciably affected. On the other hand, the validity of Ohm's law is obviously restricted by such a state of things.

The deviations will be greater the more the permeability of the ferromagnetic substance decreases—that is, the more the magnetising field increases. Assume for a moment, for the sake of argument, the complete analogy of a magnetic circuit with a voltaic pile immersed in an electrolyte (§ 112). The decrease of permeability, and its gradual approximation to unity (§ 14), would then have to find its analogue in the fact that the conductivity of the pile is in some way lessened, and continually approximates to the smaller conductivity of the surrounding liquid. The electrical current would then spread more through the electrolyte; that is, the stream-lines would leak more from the pile into the electrolyte. In accordance with this analogy, it has hitherto been generally concluded that the leakage of lines of induction must, under all circumstances, increase with the magnetising force, so that the coefficients ot leakage attain *higher* values.

This is, however, opposed to fact. We have already (§ 88) mentioned that, according to Lehmann's experiments on a uniformly-magnetised toroid divided by a radial gap, the leakage finally *diminishes*. This experiment must be regarded, in a certain sense, as a crucial one, in opposition to the view just explained. We have explained the decrease in question by Kirchhoff's law of saturation, in combination with the tangent-law of refraction of lines of induction. Electromagnetic systems might, however, be easily devised in which leakage finally *increases* with the magnetising field. This, indeed, is probably the case with most of the actual arrangements in practice, although no complete experiments on the subject are yet recorded. Everything depends on the distribution of the magnetising field of the coils with reference to the geometrical form of the ferromagnetic substance, as was previously thoroughly discussed. In the elec-

trokinetic analogue there is obviously no single feature which corresponds to these phenomena of saturation.

Our conclusion, therefore, from all this must be that it is inadmissible to apply Ohm's law to magnetic circuits. In a still higher degree is this the case with Kirchhoff's laws for branched circuits, which necessarily presuppose Ohm's law. The application of those rules to branched magnetic circuits and magnetic shunts will nearly always lead to results which are quantitatively false, and which are only correct qualitatively or as a rough approximation within a narrow range,[1] but can in no case claim a more general applicability.

§ 124. **Comparative Tables.**—Although, from the above considerations, the physical theory of the magnetic circuit is not completely identical with the theory of circuits of the most general kind, the purely formal analogy goes very far. This mathematical analogy, which extends to a number of branches of physics, is of great interest. We give, therefore, in this paragraph a comparative table from which the essential features of the analogy will be at once apparent. In six different columns the conceptions belonging to the six theories considered are stated, and in the following order:—

I. Filtration of incompressible frictionless liquids through porous bodies (Lord Kelvin).

II. Diffusion of dissolved substances in solutions (Fick).

III. Conduction of heat (Fourier).

IV. Dielectric polarisation (Maxwell).

V. Electrical conduction (Ohm).

VI. Ferromagnetic induction (Faraday, Maxwell).

In any one line of Table VII. the mutually corresponding conceptions are stated; in line

[*a*] The real or fictitious substance which flows.

[*b*] The quantity of this substance which per unit time flows through any section S; that is, the current or flux through this section.

[1] Compare Ayrton and Perry, on magnetic shunts, *Journ. Soc. Tel. Engineers*, vol. 15, p. 539, 1881. According to a private communication from the Roman physicist Pisati, who unfortunately has since then died, the irregular results obtained by him with magnetic shunts confirm what is stated in the text. So far as the author could learn, these results have not been published.

TABLE VII

	I Filtration (§ 113)	II Diffusion in Solutions	III Thermal Conduction	IV Dielectric Polarisation	V Electrical Conduction	VI Ferro-magnetic Induction
[a]	Quantity of Liquid	Mass	Quantity of Heat	—	Quantity of Electricity Q	—
[b]	Current	Mass-Current	Thermal Current	Flux of Dielectric Induction	Electric Current I	Flux of Induction $\mathfrak{G}$
[c]	Flow	Mass-Flow	Thermal Flow	Dielectric Induction	Electric Flow $\mathfrak{C}$	Induction $\mathfrak{B}$
[d]	Kinetic Energy per Unit Volume	Concentration-Gradient [2]	Temperature-Gradient	Electric Intensity	Electromotive Intensity $\mathfrak{E}$	Magnetic Intensity $\mathfrak{H}$
[e]	—	Concentration [2]	Temperature	Electric Potential	Electromotive Force E	Magnetomotive Force M
[f]	Hydrokinetic Permeability	Coefficient of Diffusion	Thermal Conductivity	Dielectric Constant	Electric Conductivity	Magnetic Permeability μ
[g]	—	—	—	—	Electric Resistivity	Magnetic Reluctivity ξ
[h]	—	—	—	—	Electric Conductance	Magnetic permeance V
[i]	—	—	—	—	Electric Resistance R	Magnetic Reluctance X

[c] The current per unit of section; that is, the flow. If this is uniform throughout the section in question, then

$$(12) \quad . \quad . \quad . \qquad [b] = [c]\,S$$

[1] Mascart and Joubert, *Electricity and Magnetism*, vol. I, § 115; London, 1883.

[2] According to the modern theory of diffusion, the osmotic gradient—or the osmotic pressure—would be introduced, which is proportional to the concentration-gradient—or to the concentration respectively. In the diffusion of gases, on the other hand, the partial pressure would have to be introduced.

[d] The agent which causes the current.

[e] The line-integral of this latter along a length L. Hence, in case of uniformity over this distance,

$$(13) \qquad . \quad . \quad . \qquad [e] = [d] L$$

[f] The property which for a given intensity of the agent [d] determines the value of the flow [c]. In all cases

$$(14) \qquad . \quad . \quad . \qquad [f] = \frac{[c]}{[d]}$$

This ratio is quite constant in (V), and, so far as at present known, also in (IV). It is approximately constant in (II) and (III), and perhaps also in (I). It is entirely variable in (VI).

[g] The property which is measured by the reciprocal of [f], and is therefore given by the equation

$$(15) \qquad . \quad . \quad . \qquad [g] = \frac{1}{[f]} = \frac{[d]}{[c]}$$

[h] By definition put equal to [f] S/L; therefore also

$$(16) \qquad [h] = \frac{[c]\ S}{[d]\ L} = \frac{[b]}{[e]} \quad \text{or} \quad [b] = [e][h]$$

[i] By definition equal to [g] L/S; therefore also

$$(17) \qquad [i] = \frac{[d]\ L}{[c]\ S} = \frac{[e]}{[b]} \quad \text{or} \quad [b] = \frac{[e]}{[i]}$$

In the last two quantities the geometrical dimensions of the body are involved, as well as its specific properties.

By merely applying the symbolical relations given to the several subjects considered we get a sufficient insight into the nature of the analogies discussed in this chapter. Table VII. comprises in this manner a large number of fundamental physical equations. By comparing the adjacent columns, V and VI, the nature of the special analogy between electric conduction and magnetic induction is seen more clearly still.

CHAPTER VIII

MAGNETIC CIRCUIT OF DYNAMOS OR ELECTROMOTORS

§ 125. **Dynamos with a Single Magnetic Circuit.**—In this and the following chapters we shall shortly explain how the results already obtained may be applied to practical problems of construction; and we shall first consider the magnetic circuit of dynamos or electromotors. Theoretically, the principle of the two kinds of machine is the same, so that, in consequence of the reversibility of their actions, any dynamo may be used as an electromotor, and conversely. In details of construction, however, the particular object aimed at has of course to be kept in view. We restrict ourselves to treating a few of the simplest and most typical cases, and for further particulars refer to special works.[1]

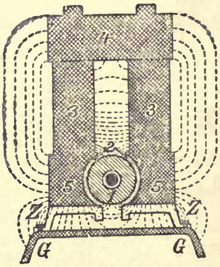

FIG. 28

In connection with Chapter VI. (§§ 96–100) we will, as an example, apply the Drs. Hopkinsons' general method there given to the simple magnetic circuit of the Edison-Hopkinson direct current dynamo, which they investigated, and which is diagramatically represented in fig. 28.[2] In this magnetic circuit we have to distinguish five principal parts, which are correspondingly numbered in the figure. The ferromagnetic parts are shaded.

1. The *Armature*.—Let the mean sectional area of the iron be S_1, the mean length which the lines of induction traverse in the iron L_1.

[1] For instance, Silv. Thompson, *Dynamo-electric Machinery*, 4th edition, 1892; Kittler, *Handbuch der Elektrotechnik*, 2nd edition, Stuttgart, 1892; Frölich, *Die dynamoelektrische Maschine*, Berlin, 1886, and many others.

[2] J. Hopkinson, *Original Papers on Dynamo Machinery and Allied Subjects*, p. 94, New York, 1893.

2. The *Interspace*—that is, the magnetically indifferent space or clearance which lies between the iron core of the armature and the cheeks of the pole-pieces, and which is occupied partly by the copper conductor of the armature, and partly by the insulating covering, and by air. Let the area of each of the two pole surfaces which enclose the armature be S_2 and the clear width of the gap L_2.

3. The *Field Magnet Limbs.*—Let the mean length of the induction lines which traverse each limb be L_3 and the mean section S_3. Let there be n turns on the two limbs together, through which the current I'_M flows.[1]

4. The *Upper Yoke.*—Let the corresponding dimensions of this part of the magnetic circuit be L_4 and S_4.

5. The two *Pole-pieces*, with the corresponding dimensions L_5 and S_5.

From the by no means simple form of the parts of the magnetic circuit, and the difficulty—at any rate, in some cases—of tracing the path of the induction lines in them, the above mean sections and lengths can only be determined with a rough approximation. These dimensions are therefore largely a matter for personal estimation.

§ 126. **Predetermination. Total Characteristic.**—In the calculation of the magnetic circuit of a dynamo (machine), which now gives results to within a few per cent., the problem, briefly expressed, is as follows :—

What value of the product (nI'_M), which is briefly called the *ampere-turns* of the field-magnets, must be chosen that there may be a prescribed value of the induction $\mathfrak{B}$ in the armature, supposed at first to be at rest and with no current flowing?

It may be remarked that a definite value for the induction cannot be prescribed for all dynamos. The best value depends, in any given case, on a number of factors. In practice, the average value of the armature-induction, which, from its normal continuity (§ 58), is almost identical with the field intensity in the air-gaps, varies between 5000 and 15,000 C.G.S. units. In dynamo machines it is unusual to exceed this latter value; on the contrary, in order to avoid the waste of energy by hysteresis

[1] It may be mentioned, the accentuation of the I' means that the current is expressed in amperes as before, and not in C.G.S. units (deca-ampere).

it is usually endeavoured to remain considerably below it (compare § 143).

The highest value mentioned must, however, still be considered small, seeing that with good wrought iron the induction $\mathfrak{B}$ may attain 60,000 C.G.S. units (fig. 3, p. 19, and Chapter IX), and that theoretically there is no limit to this quantity. The relation $\mathfrak{B} = 4\pi\mathfrak{J}$ [equation (14), § 11] is always very closely satisfied, and owing to the conditions prevailing in a dynamo machine, the permeability will always be very high, a circumstance which in many respects greatly simplifies the consideration of the case under discussion.

If we have once determined the value of the induction $\mathfrak{B}_1$ to be attained in the armature, then the total flux of induction $\mathfrak{G}_1$, which must traverse it, and therefore, for the sake of continuity (§ 61), the air-gap also, in order to have that induction in individual parts of the armature, is given by

$$\mathfrak{G}_1 = \mathfrak{B}_1 S_1$$

If now the armature, so far supposed to be at rest, is made to rotate—without, however, closing its circuit—an electromotive force E will be induced in the coils. From the principles of the induction of electromotive forces (§ 64), this is proportional to the flux of induction through the interspace, since the copper coils of the armature move through the latter. As before, we may put $\mathfrak{G}_2 = \mathfrak{G}_1$. E is further proportional to the area inclosed by the armature coils, as well as to the number of their turns. By introducing a proportional factor A, which only depends on the dimensions and the angular velocity of the armature, we may write

$$E = A\,\mathfrak{G}_2$$

The difference of potential between the two brushes of the commutator is equal to that electromotive force, so long as there is no current in the armature. For a given speed of rotation it is represented, as a function of the current I'_M in the field magnets (which at first we will consider as excited by an external independent current), by the equation

$$(1) \quad . \quad . \quad . \quad E = \text{funct.}\,(I'_M)$$

The corresponding curve is called by Marcel Deprez and J. Hopkinson the *total characteristic* of the dynamo machine for the given speed of rotation. Instead of the current, the number of ampere turns has of late been frequently introduced as the variable, so that equation (1) is then to be written

$$(1a) \quad . \quad . \quad . \quad E = \text{funct.}\ (nI'_M)$$

From what has been said above, it appears that the curves corresponding to these equations must, except as regards their scales, merely be identical with the magnetic characteristic fully discussed in § 96, the equation of which is as follows:

$$(2) \quad . \quad . \quad . \quad \mathfrak{G} = \Phi_H\,(M)$$

and which represents the flux of induction through the air-gap in relation to the magnetomotive force (§ 119) of the field magnet turns. The latter quantity is known to be represented by the equation

$$M = 0{\cdot}4\,\pi\ (nI'_M)$$

For, from the preceding considerations, the corresponding terms on the left and right sides of the equations (1) [or (1a)] and (2) differ only by the constant factors A or $0{\cdot}4\,\pi n$ (or $0{\cdot}4\,\pi$).

§ 127. **Armature yielding a Current. External Characteristic.**—As soon as a current flows through the armature, the difference of potential between the brushes—or, in other words, that between the terminals of the machine which we call E_K—is no longer equal to the whole electromotive force. It is less or greater than the latter, according as the machine produces a current or works as a motor. The curve which represents this difference of potential E_K as a function of the current in the external circuit, and which thus corresponds to the equation

$$(3) \quad . \quad . \quad . \quad E_K = \text{funct.}\ (I')$$

is called the *external characteristic* of the dynamo.

The current through the armature tends further to magnetise the latter, thereby markedly affecting the distribution in the magnetic circuit. The direction of this magnetisation is

almost at right angles to that due to the components of induction originally caused by the field magnets. The resultant total induction appears, therefore, deflected with respect of the latter component; such deflection being:

(*a*) In the direction of the rotation of the armature—that is, 'forwards'—if the machine acts as a current-generator;

(*b*) Opposite to the direction of the rotation—that is, 'backwards'—if the machine works as an electromotor.

Owing to this *armature-reaction*, which we shall discuss more in detail (§ 134), what is called the *neutral zone* experiences a forward or backward displacement, and the position of the brushes must be regulated forwards or backwards respectively, in order that the commutator may not spark. This, again, diminishes the flux of induction, owing to the armature current.

The external characteristic (3)

$$E_K = \text{funct.}\,(I')$$

for a given speed of rotation may be easily determined by means of a voltmeter and an amperemeter, and the curve then plotted. It furnishes all necessary data for the pre-determination of the output to be expected from the machine, and plays with regard to it a part like that of the indicator diagram in a steam-engine. Whether the dynamo is to act as a motor, or for generating current for various purposes, the quantities which chiefly come into consideration can always be determined, from the external characteristic, on the principles already explained. Convenient graphical constructions for this have been developed in all directions.[1]

The external characteristic, again, can be calculated from the total characteristic [equation (1)] by taking into account the above-mentioned armature-reaction, and by allowing for the resistance of the armature and of the field-magnet coils, according to the particular method of their winding. Either the current in the coils of the field-magnets or the number of their 'ampere turns' is considered as independent variable of the total characteristic (1) or of (1*a*); for the whole behaviour of the machine is really dominated by this variable. Owing to

[1] Compare Silv. Thompson, *Dynamo-electric Machinery*, 4th edition, chapter x.

the above-mentioned analogy with the magnetic characteristic $\mathfrak{G}_2 = \Phi_H(M)$, the total characteristics of all dynamo machines show an entirely similar type in their main features. But the form of the external characteristic is very materially affected by the method of winding. Thus the three principal types, 'series wound,' 'shunt wound,' and 'compound wound,' correspond to just as many kinds of external characteristic, upon the further details of which we cannot here enter.

This brief statement of the principal points of view which influence the pre-determination—that is, the estimation of the efficiency of a dynamo—lead us finally, therefore, to the determination of the total characteristic as the chief problem. And this, again, is essentially nothing more than the magnetic characteristic $\mathfrak{G}_2 = \Phi_H(M)$, the discussion of which we shall continue after the above digression.

§ 128. **Investigation of Drs. Hopkinson.**—As regards the magnetic circuit of the dynamo machine tested (fig. 28, p. 189), the following assumptions were first made in order to simplify matters:—

The total flux of induction in the upper yoke (4) is the same as that in the two limbs (3); let it therefore be called $\mathfrak{G}_3$. In like manner, the 'useful flux of induction' in the armature (1) is equal to that in the air-gap (2) and the two pole-pieces (5); let this be $\mathfrak{G}_2$.

These two values of the flux of induction are now unequal, and their ratio is defined by the Hopkinsons (p. 82) as the leakage-coefficient,

$$\nu = \frac{\mathfrak{G}_3}{\mathfrak{G}_2}$$

Starting now from the flux of induction in the air-gap, which, from what has been said, determines the electromotive force to be produced, the general equation (III) of § 100 will be in the present case

$$\text{(I)} \qquad . \quad . \quad . \qquad M = 0{\cdot}4\,\pi\,n\,I_M$$

$$= L_1 f_1\left(\frac{\mathfrak{G}_2}{S_1}\right) + 2L_2\frac{\mathfrak{G}_2}{S_2} + 2L_3 f_3\left(\frac{\nu\,\mathfrak{G}_2}{S_3}\right) + L_4 f_4\left(\frac{\nu\,\mathfrak{G}_2}{S_4}\right)$$

$$+ 2L_5 f_5\left(\frac{\mathfrak{G}_2}{S_5}\right)$$

in which M is the total magnetomotive force produced by the field coils (§ 119), and f_1, f_3, f_4, f_5 denote the functions

$$\mathfrak{H} = f(\mathfrak{B})$$

for the corresponding ferromagnetic parts which form the magnetic circuit (*cf.* § 96).

The above equation (I) was experimentally tested by Drs. Hopkinson on the machine which has frequently been mentioned. The pole-pieces were separated from the cast-iron base-plate G by a magnetically indifferent piece Z of cast zinc. The principal dimensions of the machine may be estimated from the statement that fig. 28, p. 189, is on a scale of $\frac{1}{40}$ full size. The following data may be given in addition:—

1. The armature consisted of 1000 iron discs separated by sheets of paper and strongly pressed together. The section S_1 was taken as $= 810$ sq. cm., the distance $L_1 = 13$ cm. The winding was on the von Hefner-Alteneck system, and had 40 turns, each with a resistance of nearly 0·01 ohm. With a normal speed of 12·5 revolutions per second, the armature gave 320 amperes at 105 volts. The flux of induction $\mathfrak{G}_1$ then amounted to 11 millions C.G.S. units, corresponding to an induction $\mathfrak{B}_1 = \mathfrak{G}_1/S_1 = 13{,}500$ C.G.S.

2. Air-gap: cross-section $S_2 = 1600$ sq. cm.; width $L_2 =$ 1·5 cm.

3. The limbs consisted of hammered and afterwards annealed wrought iron. The rectangular cross-section was $22 \times 44{\cdot}5$ sq. cm. For convenience of winding, the corners were somewhat rounded off, so that $S_3 = 980$ square cm. The length of each limb was $L_3 = 46$ cm. The total number of turns on each limb was $n = 3260$.

4. Yoke: $S_4 = 1120$ sq. cm.; $L_4 = 49$ cm.

5. Pole-pieces: $S_5 = 1230$ sq. cm.; $L_5 = 11$ cm.

The functions f_1, f_3, f_4, f_5 were considered as identical, and taken from curves of induction previously determined by J. Hopkinson. The leakage-coefficient was experimentally determined in the manner to be afterwards described, and in accordance with this ν was put equal to 1·32 (§ 130).

§ 129. **Graphical Construction.**—In the previous paragraphs all the necessary data are given, which are now simply to be

inserted in equation (I). According to the practice of Drs. Hopkinson, it is usual to make the calculation in question by the graphical method, as we have already done in the comparatively simple case of a toroid divided radially (fig. 25, p. 151).

The graphical construction for the Edison-Hopkinson dynamo just described is represented in fig. 29, p. 197, which is taken directly from Drs. Hopkinson's paper. The abscissæ represent the magnetomotive force M, the ordinates the flux of induction through the air-gap.[1]

The curve (H) corresponds to the pole-pieces, (A) to the armature, (G) to the yoke, (C) the limbs of the magnet, the straight line (B) to the air-gap. The latter claims, as is obvious—especially for small values of the induction—by far the greater part of the total magnetomotive force.

From the five partial curves which correspond to the five members of equation (I), the resultant curve (D) is ultimately found by adding the abscissæ. (C) and (G), moreover, are not simple curves, but loops corresponding to hysteresis of the ferromagnetic parts in question. The ascending or descending branches are denoted by arrows. This, on addition, is also transferred to curve (D), which thus appears as a loop, and represents $\mathfrak{G}_2 = \Phi_H(M)$ for the whole magnetic circuit of the machine.

In the experiments made to test the theory, the field-magnets were excited by a current I'_M, separately generated and measured. From this there was found, by means of the number of turns mentioned,

$$M = 0{\cdot}4\,\pi \times 3260\,I'_M = 4100\,I'_M$$

The corresponding difference of potential E_K at the terminals, for the normal speed of rotation and no current in the armature, was determined; it was therefore equal to the electromotive force. It was plotted on a suitable scale as a function of M.

[1] The numbers on the scale of abscissæ represent (as with Hopkinson) C.G.S. units; multiplied by 0·8 (which is very nearly equal to $1/0{\cdot}4\pi$); this gives M in ampere-turns, as is also usual. The numbers of the scale of ordinates represent millions of C.G.S. units.

The highest values of M or of $\mathfrak{G}$ in this machine were about as follows:

M: 50,000 C.G.S. = 40,000 ampere-turns. $\mathfrak{G}$: 15×10^6 C.G.S. units.

The points thus obtained are denoted for ascending currents by +, and for descending by ⊕, and are also entered in fig. 29.

The agreement, as will be seen, is as good as can at all be expected with such measurements on a working machine. It is true that the points observed for higher magnetomotive forces lie somewhat below the theoretical curve (§ 132). The approximate correctness of the theory, and the utility of the graphical construction depending on it, are, however, on the whole, confirmed by the experiments.

FIG. 29

§ 130. **Experimental Determination of the Leakage.**—The coefficient of leakage ν was determined by Drs. Hopkinson by special experiments with exploring coils (§ 2).

In the first case, a single turn of wire was passed round the middle of the limb of the magnet, the ends of the wire being connected with a ballistic galvanometer. A short circuit was then connected with the field coils, on opening or closing which a throw was observed in the galvanometer.[1] This measured the

[1] The general objection may indeed be raised against such ballistic measurements on dynamo machines, &c., that the period of the galvanometer can scarcely be assumed to be long enough as compared with the very long time which, owing to their considerable self-induction, the limbs required for their magnetisation, demagnetisation, or reversal of magnetisation. That assumption is, however,

total flux of induction through the limbs, in so far as residual induction may be disregarded.

In the second case, the throw of the ballistic galvanometer was determined when its leads were soldered to those segments of the commutator, corresponding to that armature winding which lies in what is called the plane of commutation. In the machine examined, fig. 28, this is the vertical plane. This throw gave a measure of the 'useful flux of induction' through the armature (§ 128).

The ratio of the first throw to the second, is also that of the total flux of induction to the useful flux—that is, it is equal to the leakage-coefficient, whose value for the machine in question was thus found to be 1·32. The leakage-coefficient was here found to be almost constant—that is, independent of the value of the induction—which is not surprising considering the small value of the latter (§§ 95, 126).

If, therefore, we put the total flow of induction equal to 100, the effective flux through the armature becomes $100/1{\cdot}32 = 75{\cdot}8$. The difference 24·2 is equal to the 'stray field.' The question now arises how this latter is distributed in the various air-spaces. Drs. Hopkinson have investigated this question also by placing loops of wire in various places and directions, and observing as above, the induced throws with the same ballistic galvanometer. They thus found:

1. Stray field:	
Between the pole-horns	2·8
,, ,, limbs	7·0
Through the cast-iron base-plate .	10·3
Other leakage	4·1
2. Useful induction	75·8
Total flux of induction . . .	100·0

In fig. 28, p. 189, the dotted lines of intensity give, according to Silv. Thompson, an approximate idea of the leakage in the entire air-space. The base-plate, notwithstanding that a

an essential condition for applying the ballistic method. Moreover, in the present case it would have been better to wind the test coil round the upper part of the limb than about the middle. (See the calculations of Forbes, § 132.)

layer of zinc Z is interposed, plays an unfavourable part, as it partly forms a magnetic short circuit.

Similar very complete measurements have been made by Lahmayer.[1] Experiments on leakage have further been published by Hering, Carhart, Trotter, Esson, Corsepius, Wedding, &c.,[2] into the details of which we cannot enter here, since they can hardly claim a general theoretical interest, but hold rather for special cases—that is, special types of machine—and are therefore chiefly valuable from the practical point of view.[3]

§ 131. **Introduction of Magnetic Reluctance.**—We will now consider the magnetic circuit of dynamo machines from the point of view of the considerations developed in Chapter VII. For this we introduce magnetic reluctance, and again divide the magnetic circuit into five parts, as above (§ 125). The total magnetomotive force is thus the sum of five parts, each of which may be regarded as the product of the flux of induction in question, into the corresponding magnetic reluctance, and therefore takes a form defined by equation I (§ 119). The sum of the five terms—that is, the right side of the Hopkinson's equation (I) § 128—thereby at once becomes susceptible of another interpretation. This transformation of the Hopkinson's equation, which we alluded to in § 122, is a purely formal one, which does not at all affect the essence of the matter, but is, in a certain sense, already contained in it. There is no contradiction between Hopkinson's treatment and that to be now discussed. In order to explain this analytically, we observe that each of the five terms of equation (I), § 128—for instance, the first—by remembering the definitions of § 119, may be transformed as follows:

$$(4) \qquad . \qquad L_1 f_1 \left(\frac{\mathfrak{G}_2}{S_1}\right) = L_1 \mathfrak{H}_1 = M_1 = X_1 \mathfrak{G}_2$$

[1] Lahmeyer, *Elektrotechn. Zeitschrift*, vol. **9**, p. 283, 1887.

[2] C. Hering, *Electr. Review*, vol. **21**, pp. 186, 205, 1887; Carhart, *Electrician*, vol. **23**, p. 644, 1889; Trotter, *Journ. Inst. Electr. Engineers*, vol. **19**, p. 243, 1890; Esson, *ibid.* p. 122, 1880; Corsepius, *Theoretische und praktische Untersuchungen zur Konstruktion magnetischer Maschinen*, p. 85 *et sqq.*, Berlin, 1891; Wedding, *Elektrotech. Zeitschrift*, vol. **13**, p. 67, 1892.

[3] Kittler, *Handbuch der Elektrotechnik*, 2nd edition, vol. **1**, p. 650, Stuttgart, 1892, gives a very complete synoptical table of the results of experiments on leakage.

By transforming each individual term of equation (I) in a similar manner, the complete expression of the magnetic characteristic becomes

$$(\mathrm{I}a) \qquad M = X_1 \mathfrak{G}_2 + 2 X_2 \mathfrak{G}_2 + 2 X_3 \nu \mathfrak{G}_2 + X_4 \nu \mathfrak{G}_2 + 2 X_5 \mathfrak{G}_2 = F_H(\mathfrak{G}_2)$$

or

$$(\mathrm{II}) \quad \mathfrak{G}_2 = \frac{M}{2 X_2 + (X_1 + 2 \nu X_3 + \nu X_4 + 2 X_5)} = \Phi_H(M)$$

The flux of induction through the air-gap is therefore equal to the total magnetomotive force, divided by the sum of the magnetic reluctances of the parts of the magnetic circuit. The leakage is allowed for by multiplying the resistances of the limbs and of the yoke by the leakage-coefficient. The fraction of the magnetomotive force requisite for magnetising these parts is, in fact, not greater because its resistance had increased; but of course this is only the case because a greater flux of induction $\mathfrak{G}_3 = \nu \mathfrak{G}_2$ must pass through it. In order to explain this, we may write equation (II) differently, by introducing the total flow of induction $\mathfrak{G}_3$, instead of the useful flux $\mathfrak{G}_2$:

$$(\mathrm{II}a) \quad \mathfrak{G}_3 = \nu \mathfrak{G}_2 = \frac{M}{2 \frac{X_2}{\nu} + \left(\frac{X_1}{\nu} + 2 X_3 + X_4 + 2 \frac{X_5}{\nu}\right)}$$

by which the influence of leakage becomes clear enough, since it apparently diminishes the reluctances X_1, X_2, and X_5.

Nothing can be objected to equation (II) or to (IIa) as long as we keep in view that all reluctances are variable except X_2, the constant reluctance of the air-gap. Although, theoretically, the mode of writing equation (II), as compared with the original form of equation (I), offers scarcely any advantage, there is, in practice, one circumstance which places the question in an essentially different light.

The constant reluctance X_2 of the air, and of the copper between the armature and the pole-pieces, is, especially in dynamos, almost always considerably greater than all the other variable reluctances together—at any rate, within the small range of induction ($\mathfrak{B} < 15{,}000$ C.G.S.) which occurs in practice—so that this portion of the magnetic circuit requires

by far the greater part of the whole magnetomotive force. We mentioned this already in § 129 in considering fig. 29, in which the straight air-line (B) mainly determines the form of the characteristic (D). We may therefore assume somewhat arbitrary values for the permeability of the main portions of the magnetic circuit, without the result being materially affected, provided the reluctance of the air is correctly brought into the calculation. It is just this, however, which presents special difficulties, and, according to the judgment of experienced electrical engineers, is frequently a weak point in the application of the foregoing theory. We will now enter more closely into the theoretical and empirical determinations of air-reluctances.

§ 132. **Calculation of Air-reluctances.**—So long as it is only a question of the layer of air between two parallel ferromagnetic surfaces of area S, at a distance d, the magnetic reluctance, in accordance with the theoretical definition, is simply

$$(5) \quad . \quad . \quad . \quad . \quad X = \frac{d}{S}$$

or the magnetic permeance

$$(6) \quad . \quad . \quad . \quad . \quad V = \frac{S}{d}$$

since the permeability of air is equal to unity. In many cases, however, the calculation is not so simple, and the magnetic permeance of air-spaces has to be determined between surfaces of iron of arbitrary shape.

Forbes[1] has approximately calculated the permeance for a few special cases of most frequent occurrence by making simple assumptions as to the path of the lines of induction in the air, and has given the following solutions:—

I. Magnetic permeance of the gap between two parallel surfaces S_1 and S_2, unequal, but not very much so (fig. 30, p 202). If it be assumed that the lines of induction are straight and uniformly distributed over the two surfaces, then obviously,

$$(7) \quad . \quad . \quad . \quad V = \frac{S_1 + S_2}{2\,d}$$

[1] Forbes, *Journ. Soc. Tel. Engineers*, vol. **15**, p. 551, 1886.

II. Magnetic permeance of the space between two equal rectangular ferromagnetic surfaces, which lie near each other in one plane, as represented in fig. 31. Assuming that the lines of induction are semicircles, as shown in the figure by the dotted lines, Forbes finds:

$$(8) \quad . \qquad V = a\int_{r_1}^{r_2} \frac{dr}{\pi r} = \frac{a}{\pi}\log_e \frac{r_2}{r_1} = \frac{a}{\pi}\log_e \frac{d_2}{d_1}$$

in which $d_1 = 2r_1$, $d_2 = 2r_2$, and the meaning of the other letters follows from fig. 31.

III. Magnetic permeance in the case represented by fig. 32, where the ferromagnetic surfaces are further apart. Assuming

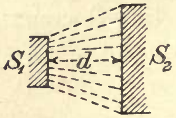

FIG. 30

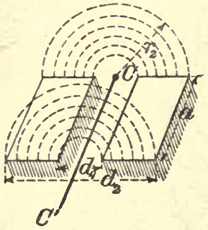

FIG. 31

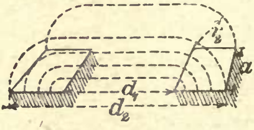

FIG. 32

that the path of the lines of induction is as represented in the figure, Forbes finds:

$$(9) \quad . \qquad V = a\int_{0} \frac{dr}{\pi r + d_1} = \frac{a}{\pi}\log_e \frac{\pi r_2 + d_1}{d_1}$$

If the two surfaces do not enclose the angle π, as in case II (fig. 31), but an angle a, in which case they are supposed to be rotated about the right line $\overline{CC'}$, then the value of a in circular measure must be substituted for π in equation (8).

In these calculations there is much of an arbitrary nature in the simple assumptions made, that induction lines consist of arcs of circles and straight lines; in such instances, however, success alone can decide. In practical application the above formulas have in many cases been approximately confirmed, inasmuch as with their aid it is often possible to obtain an

almost exact estimate of the leakage of a machine before its construction. Forbes had already calculated the leakage of the Edison-Hopkinson dynamo described above (§§ 125–130) from the leakage-diagram represented in fig. 28, p. 189; from this he calculated a coefficient of leakage $\nu = 1{\cdot}40$, independent of the value of the induction. He showed, further, that if this value is inserted in equation (I), § 128, the curve D (fig. 29, p. 197) agrees still better with the points observed than by introducing the value $\nu = 1{\cdot}32$, as measured by Drs. Hopkinson (§ 120). The cause of this discrepancy is probably partly to be sought in the fact that they placed the exploring coil around the middle instead of around the top of the limbs of the magnet (compare note, p. 197).

On the other hand, it is not to be denied that in many other cases the theoretical computation of magnetic air-reluctances leaves much to be desired, since very often considerably smaller reluctances were observed than had been calculated. This is due, in great part, to the fact that at the edge of the air-gap the lines of induction, owing to leakage, bend outwards. The section of the air-gap to be considered in the calculation is thereby greater than the geometrical one, and to an extent which can scarcely be correctly estimated otherwise than empirically. In their fundamental research the Hopkinsons point out this circumstance, and allow for it by putting the section of the air-gap $S_2 = 1600$ sq. cm. (as given in § 128), while it amounted, in fact, only to 1410 sq. cm.

§ 133. **Other Determinations of Air-reluctances.** — From the special reason, so frequently insisted on, that in dynamo machines the constant magnetic reluctances of the air-spaces play the chief part, the analogies discussed in the previous chapter may be used to determine or to approximately estimate them. The permeability of the air-space is unity, while that of the ferromagnetic substance may be regarded as infinitely great in comparison, since, under the conditions existing in a dynamo it has always very high values (§ 126).

Let us now consider the dielectric analogy (Table VII, column IV, § 124). The magnetic permeance of an air-space of any given form is, in consequence of this analogy, there shown to be proportional to its electrostatic capacity, in so far

as the ferromagnetic surfaces may be regarded as the conducting plates of a condenser. Hence, by experimentally determining the latter, the former might be found.[1]

Let us then further consider the analogy with the conduction of electricity (*loc. cit.*, column V), which deserves the preference in practically determining the magnetic permeance of air-spaces. The latter quantity is proportional to the electrical conductivity of a badly conducting electrolyte, which surrounds well-conducting electrodes, the form of which, again, corresponds to that of ferromagnetic bounding surfaces.

In this connection we may refer to the observations of Ayrton and Perry.[2]

As regards the determination of leakage by a direct magnetic method, we refer, finally, to Lehmann's empirical leakage formula, which holds for two semi-infinite cylinders which are at a certain distance from each other (§ 90). In dynamo machines it will be useful only in exceptional cases. We shall recur to its application in the following chapter.

§ 134. **Influence of the Position of the Brushes.**—A consideration of the magnetic circuit of dynamos or electromotors would be incomplete without a more detailed discussion of the armature-reaction already mentioned in § 127. In their pioneering investigation the Drs. Hopkinson take this question also into consideration.[3] We shall, in what follows, reproduce

[1] Du Bois, *Wied. Ann.* vol. **46**, p. 495, 1892. Since the dielectric constant of the air is 1, the factor of proportionality will be 4π, as appears from the equations of electrostatic capacity.

[2] Ayrton and Perry, 'The Magnetic Circuit of Dynamo Machines,' *Phil. Mag.* [5], vol. **25**, p. 505, 1888. One of the best ways of finding the nature of the probable magnetic leakage in a dynamo before it is constructed, is to construct a small model of the same kind of iron, exciting the field magnets, and exploring by means of a ballistic galvanometer. Another simpler method which gives a considerable amount of information, and which we have employed, is to make a model of wood, covering certain judiciously selected parts such as the poles and armatures and half the field magnets with metal, immerse the model in a barrel of rainwater and find the electric resistance between one part and another when electric potential-differences are established between them. On account of the ease with which the model may be rearranged in configuration, this method of working gives interesting results; but these results when applied in the magnetic case must be used judiciously, and with the knowledge that the permeability of iron is not a fixed quantity.

[3] J. and E. Hopkinson, *Phil. Trans.* vol. **177** [I], pp. 342–347, 1886. *On Original Papers on Dynamo Machinery and Allied Subjects*, pp. 103–111, New

the considerations referred to (*loc. cit.*), without many alterations, and rather fully, as they furnish a suitable example of the practical application of the theory of the magnetic circuit.

We have already pointed out (§ 127) that besides the current in the coils of the field magnets I'_M assumed as the single independent variable, the armature current I'_A has also an influence on the direction[1] and, indirectly, on the value of the flux of induction, so that it must be brought in as a second independent variable. The electromotive force of the machine will therefore, strictly speaking, be represented by a surface instead of by a curve. In the more recent, well-constructed machines the influence of the armature current is, indeed, reduced to a minimum, so that it generally exerts not more than 5 per cent. of the action of the currents in the field magnets, but it is nevertheless not to be disregarded. In the older machines it was sometimes as much as 25 per cent.

If one segment of the armature coils is commutated, it must be short-circuited for a moment. If during this time the variation of the flux of induction encircled by the coils in question is not very small, a powerful current is produced, which causes a loss of output and objectionable sparking. The most suitable position of the brushes is that in which, during the short circuiting of a segment of the coil, the flux of induction $\mathfrak{G}$ encircled by this just passes through a maximum. At the beginning of the short circuiting $\mathfrak{G}$ increases somewhat, and a

York, 1893; this latter reprint greatly facilitates the study of the Hopkinson researches in their original form, which is highly to be recommended.

[1] It was formerly usual to ascribe the deflection of the direction of magnetisation in the armature to a certain dragging of the lines of induction by the rotation of the armature; this was then explained by the occurrence of what was called magnetic lag, with regard to which our knowledge is still very incomplete (Wiedemann, *Elektricität*, vol. **4**, §§ 209–326; Ewing, *Magnetic Induction*, §§ 88–89.) That deflection may also partly be due to *directional hysteresis*, a phenomenon on which there is at present very scanty information; it may be conceived as a more general case of ordinary hysteresis, in which, besides the numerical value and the sign of the two magnetic vectors, their direction also comes into account, which, in investigations on hysteresis is tacitly assumed to be invariably the same [compare Baily, *The Electrician*, vol. **33**, p, 516, 1894; H. d. B.]. Although there now can be no doubt that practically the armature current plays by far the chief part, yet a certain influence, even if it be a small one, must always be ascribed to those two factors, as well as to the occurrence of eddy currents (§ 143), which will always produce a deflection of the direction of induction.

weak current is induced in the previous direction. Then follows the maximum itself, where the variation is infinitely small, and therewith the induced current vanishes. Finally, $\mathfrak{G}$ somewhat decreases, so that a weak current now passes in the opposite direction.

The best position of the brushes depends on a number of circumstances. In practice that position is adopted which yields the minimum of sparking. We shall therefore consider it as an independent variable which is exclusively determined by the attendant. The measure of the displacement of the brushes is the angle a which the neutral zone makes in the direction of rotation of the armature with its position of symmetry—that is, with the 'plane of commutation' with no current in the armature. The angle a is therefore positive if the machine acts as a current-generator with a forward lead of the brushes, negative if used as motor with a backward displacement. We shall in the sequel consider it as expressed in circular measure (that is, if γ is the same angle in degrees, then $a = \pi\gamma/180$).

§ 135. **Calculation of the Armature-Reaction.**—Let the armature current I'_A be reckoned positive in the direction of the electromotive force induced in the machine itself. If, therefore, it acts as a generator, the current is positive; if as a motor, it is negative. Let us now consider any closed path of integration through the magnetic circuit of the machine $\overline{ABCDEFA}$ (fig. 33), for instance, and first apart from the loop $\overline{BCGH}$. The line integral $M = 0{\cdot}4\,\pi\,n\,I'_M$ of the intensity produced by the field magnets (§ 126) is then, if the machine produces current, diminished owing to the action of those ampere-turns of the armature,

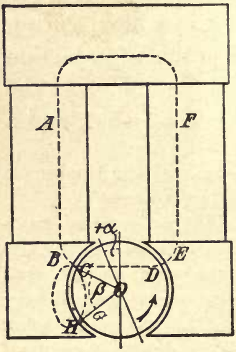

Fig. 33

which lie between the planes making the acute angle $+\,a$ and $-\,a$ with the plane of symmetry (in this case the vertical plane). The other ampere-turns of the armature produce lines of induction parallel to the (vertical) plane of symmetry, and only, therefore, affect the direction of the flux of induction, and not its value.

With an Edison-Hopkinson armature of m turns it may be shown, from the manner in which the curve of integration is involved with the windings, that this diminution of the line-integral—that is, in a certain sense, the 'counter-magnetomotive force'[1] of the armature—is equal to $0{\cdot}4\,a\,m\,I'_A$, in which, as has been observed, a is expressed in circular measure. Hence, with an armature producing a current the total electromotive force will be

$$(10) \quad . \quad . \quad M = 0{\cdot}4\pi n I'_M - 0{\cdot}4\,a m I'_A$$

With a series machine, in which the current of the armatures also flows through the field magnets, and therefore $I'_M = I'_A$, we have

$$(11) \quad . \quad . \quad M = 0{\cdot}4\,I'_A\,(\pi n - am)$$

Besides this diminution of the total magnetomotive force, a displacement of the distribution of the lines of induction over the hollowed-out pole-pieces occurs, owing to the component of induction due to the rest of the windings of the armature. Thus, for instance, the induction at $\overline{BC}$ is not the same as at $\overline{GH}$ (fig. 33). For considering the loop $\overline{BCGH}$ as a path of integration, the parts of the line-integral along $\overline{CG}$ and $\overline{BH}$ are to be neglected. The total line-integral will be determined by the ampere-turns of the armature windings encircled by the path of integration and corresponding to $\overline{CG}$. This is obviously equal to the difference ΔM of the magnetomotive forces between B and C on the one hand, and between G and H on the other; for if both these parts were equal, their algebraical sum—that is, the whole line-integral—would be zero, which is not the case. If the angle COG is denoted by β (in circular measure), it may be shown that

$$(12) \quad . \quad . \quad . \quad \Delta M = 2\beta m I'_A$$

[1] It is to be observed that with a forward lead of the brushes the magnetomotive force of the armature coils forms a 'counter-action,' if the machine generates current; but, on the other hand, assists that of the field magnets if it acts as motor; with a backward displacement of the brushes, precisely the reverse is the case. From what has been said about the position of the brushes, as defined by the position of least sparking, it is clear that in practice a counter-magnetomotive force will always occur. (Compare Silv. Thompson, *loc. cit.* pp. 585–590.)

This disturbance of the distribution remains without essential influence on the total difference of potential between the brushes, but materially influences the distribution of potential round the commutator. This, as we know, is represented in the ideal case of one turn rotating uniformly in a uniform field, by a sine curve, which, however, is more or less distorted, owing to the armature-reaction.

§ 136. **Experiments on Reactions of the Armatures.**—In a recent Memoir of J. Hopkinson and Wilson[1] the calculations of the preceding articles have been tested on two machines of Siemens Brothers, which were investigated both as dynamos and as motors. In the first place, the distribution of potential around the commutator was determined by means of an insulated auxiliary pair of brushes, capable of rotation, and the curves plotted. These show the characteristic distorting influence of the reaction of the armature already referred to, which differs according as the machine is used as a dynamo or as a motor. An agreement was observed with the above theory which must be considered as satisfactory, having regard to the numerous sources of error in such experiments.

In the machines investigated, the armature had a larger section S_1 than the limbs of the magnet, so that the first member of the Hopkinson equation [(I), § 128] could be neglected. Only the air-gap (L_2, S_2), the pole-pieces, limbs, and yoke were taken into account. The latter had the total length L_3 and mean section S_3. The equation is thus simplified, and, apart at first from armature-reaction, has the following form:

$$(13) \qquad . \qquad 0{\cdot}4\pi n I'_M = 2 L_2 \frac{\mathfrak{G}_2}{S_2} + L_3 f\left(\frac{\nu \mathfrak{G}_2}{S_3}\right)$$

which, by introducing the Hopkinson function Φ_H of § 96, we can abbreviate as follows:

$$(14) \qquad . \qquad \mathfrak{G}_2 = \Phi_H \,(0{\cdot}4\pi n I'_M = \Phi_H\,(M)$$

Φ_H is thus the magnetic characteristic for the armature current $I'_A = 0$. In Drs. Hopkinsons' first paper a general equation

[1] J. Hopkinson and Wilson, *Proc. Roy. Soc.* Feb. 1892; *Electrician*, vol. **28**, p. 609, 1892. Reprint, pp. 136–147, New York, 1893.

has already been deduced, in which the armature-reaction was allowed for. By introducing the function Φ_H it became (*loc. cit.*, p. 108):

$$(15) \quad . \quad \mathfrak{G}_2 = \Phi_H \left(0{\cdot}4\,\pi\, n I'_M - \frac{4\,a\,m\,I'_A}{\nu}\right) - \frac{2\,(\nu - 1)}{\nu} \cdot \frac{a\,m\,I'_A S_2}{L_2}$$

The letters have the same meaning as before. It was remarked at the outset that in this equation $\mathfrak{G}_2$ appears as a function of the two independent variable currents I'_M and I'_A, and, strictly speaking, can only be represented graphically by a surface. This 'characteristic surface' was geometrically discussed in the place referred to. From this was more especially deduced the ordinary two-dimensional characteristic of a series machine, for which the two variables $I'_M = I'_A$ coincide. Among others, it was shown that in certain conditions the flux of induction through the armature could attain a maximum, and then decrease, in accordance with the known fact that in series machines the electromotive force often attains a maximum, and afterwards considerably decreases.

The somewhat complex equation (15) was tested by Hopkinson and Wilson on the above machines, and found to be confirmed within certain limits. In this one machine was used as a dynamo and the other as a motor, the field magnets being separately excited by a battery. For more details we must refer to the original, as well as for further investigations on armature-reaction, which are of less interest for the theory of the magnetic circuit; we must be content with a reference to the works cited in the note on p. 189.

§ 137. **Empirical Formulæ.**—We have shown in the preceding how from the *normal* curves of magnetisation or of induction (*i.e.* for those which hold for endless shapes) we can obtain in a rational, though only approximate manner, the relation which exists between the useful flux of induction through the air-gap, and the total magnetomotive force of a dynamo. The methods developed may be best exhibited and worked out graphically; we have hitherto made no attempt to obtain an analytical expression for the curves.

Such expressions for magnetic curves have long been sought, though it is not easy to see why an approximately correct empirical formula should better satisfy the human tendency to enquiry, or be better fitted for solving practical problems, than a curve which is also empirical, but which at least represents the actual processes exactly as they have been observed. In the course of time a great number of the most varied empirical formulæ have been proposed, often without any regard as to whether they held for perfect or for imperfect magnetic circuits, with more or less extensive air-spaces intervening. The latter circumstance, however, constitutes a fundamental difference, and, from what has been said, needs no further comment.

In § 118 we have already mentioned the tan [1] formulæ which Kapp proposed, after the precedent of J. Müller and von Waltenhofen, and applied to the magnetic circuit of a dynamo. For a critical historical account of all the formulæ proposed, we must be content with a reference to special works, as they can only claim historical interest.[1] One formula, already frequently mentioned (§§ 33, 121), we will, however, consider more closely; this corresponds to O. Frölich's so-called 'curve of active magnetism.'

§ 138. **Frölich's Formula.**—This formula was established empirically by its inventor, as giving the simplest analytical representation of the curves observed with dynamo machines. It has certain points of connection with a first approximation formerly given by Lamont[2] to the expansion in series of an exponential formula. Frölich's formula forms the basis of his own theory of dynamo machines,[3] as well as of that of Clausius.[4]

This formula, according to Frölich (*loc. cit.*, p. 11), is 'the same for all machines, and the only individual feature is the scale on which the abscissæ are plotted'; and we may, there-

[1] G. Wiedemann, *Elektricität*, vol. **3**, § 450, *et seq.* 1883; Silv. Thompson, *Dynamo-Elect. Machinery*, 3rd edition, pp. 302–311, 1888.

[2] Lamont, *Handbuch des Magnetismus*, p. 41, 1867.

[3] O. Frölich, *Die dynamoelektrische Maschine*, 1886.

[4] Clausius, 'Zur Theorie der dynamoelektrischen Maschinen,' *Wied. Ann.* vol. **20**, p. 353, vol. **21**, p. 385, vol. **31**, p. 302.

fore, represent it by as general an equation as possible, as follows:

$$(16) \qquad . \qquad y = \frac{\frac{1}{Q}x}{1 + \frac{1}{Q}x} \quad \text{or} \quad x = \frac{Qy}{1-y}$$

In this x is proportional to the 'ampere-turns' of the field magnet, y to the useful or 'effective' flux of induction through the air-gap. The latter would from the above formula attain an asymptotic maximum value for $x = \infty$, which, strictly speaking, does not hold for induction (fig. 3, p. 19); of course, the formula only claims approximate validity within the range used in practice. That assumed maximum value is at the same time chosen as unit of ordinates, for it appears from the formula that

$$\text{for } x = \infty,\ y = 1$$

On the other hand,

$$\text{for } x = Q,\ y = \tfrac{1}{2}$$

Q is therefore that value of x for which the iron is 'half saturated,' or, as Silv. Thompson[1] has appropriately expressed it, has attained its 'diacritical point.'

Frölich's empirical formula, as is at once seen, represents a hyperbola, one asymptote of which runs parallel to the x-axis at a distance 1.

§ 139. **Relation of Frölich's to other Formulæ.**—There is a certain interest in comparing this curve of 'active magnetism'[2] with the curve of magnetisation deduced by the author in a rational, though only approximate, manner for bodies with a high factor of demagnetisation. A hyperbola was also here found, one asymptote of which was identical with the preceding; its equation [§ 33, equation (18)] was

$$(17) \qquad x = Ny + \frac{P}{1-y} \quad \text{or} \quad x = \frac{Ny + (P - Ny^2)}{1-y}$$

Here N is the factor of demagnetisation, P a second constant which depends on the nature of the ferromagnetic

[1] Silv. Thompson, *Dynamoelectric Machinery*, 3rd edition, p. 307, 1888.

[2] Du Bois, *Verhand. der Sections-Sitz. des Elektrotechn. Kongress*, Frankfurt, p. 75, 1891.

substance; the numerical value of the maximum magnetisation is to be regarded as unit of co-ordinates, both for the ordinates and for the abscissæ.

The second way of writing the two equations (16) and (17) shows that they differ, by the expression $(P - Ny^2)$ in the numerator. The second asymptote of Frölich's hyperbola differs from that of the curve last mentioned; the former, too, is somewhat flatter than the latter, and it is never possible to give such a value to Q that both curves coincide.

In this comparison of the two curves obtained in different ways, it must be borne in mind that, for the sake of better comparison, the formulas have been brought into the simple form $y =$ funct. (x) or $x =$ funct. (y), without taking into account the physical meaning of the co-ordinates. Now, apart from factors of proportionality, which only affect the scale of co-ordinates, Frölich's formula (16) represents a magnetic characteristic (§ 96) that is a relation between the actual flux of induction $\mathfrak{G}$, and the total magnetomotive force M; equation (17), on the contrary, represents a curve of magnetisation (§ 13) different from the normal curve—that is, a relation between magnetisation $\mathfrak{J}$ and intensity $\mathfrak{H}$ in an imperfect magnetic circuit. We have, however, shown on a former occasion how two such apparently different functions may without difficulty be transformed into each other (§ 98).

In the original form, Frölich's formula (16) had also a term in x^2 in the denominator; this was subsequently omitted, as it was found to be superfluous, since the simpler form could equally well represent the case. This latter formed for many years the only guide for interpreting many obscure points which the action of dynamos then presented. The value of so simple a formula with which the complicated action of a machine could be represented, even empirically and approximately, was not to be under-estimated for practical requirements.

The connection of the empirical linear equation (10) § 121 for the magnetic reluctivity ξ, as function of the intensity $\mathfrak{H}$, with Frölich's formula we have already discussed; it strikes one at first sight, if we compare equation (11) p. 207 with equation (16) p. 211. Of course, the latter then only

applies to closed magnetic circuits without air-gaps, and holds exclusively for the range within which we may write

$$\xi = a + b\mathfrak{H}$$

for the ferromagnetic substance in question.

In accordance with this Frölich thus expresses himself in a recent paper[1]:—'My older formula (16), as appears from our considerations, is valid for electromagnets with small air-spaces, and therefore precisely for the newer dynamos in which the air-spaces are reduced to a minimum.' In connection with this Frölich then develops a newer formula with a member added, which holds also with a certain approximation for electromagnets with interspaces of any given extent.

But since equation (17), as expressly observed, only holds for imperfect magnetic circuits with a considerable demagnetisation-factor, its want of agreement with Frölich's older formula need not cause surprise, its justification being found rather in the very different range of applicability of the two expressions.

§ 140. **General Arrangement of the Magnetic Circuit.**—Hitherto we have always elucidated our discussion (mainly theoretical) of the magnetic circuit of dynamo machines and electromotors, by reference to an example which has become classical, the Edison-Hopkinson machine (fig. 28, p. 189). We now proceed to discuss the very varied forms which such magnetic circuits present in the machines occurring in practice.

Notwithstanding the extraordinary variety in the types of construction which have been developed in the designing of dynamos and electromotors during the last twenty years, and which have in part established themselves, and the corresponding difference in the arrangement of the magnetic circuit, three main parts may always be discriminated in the latter.

1. *The Field-magnet*, the object of which may be stated in the most general terms to be, the permanent conduction of the flux of magnetic induction of the machine for the greater part of its course; at the same time, in the great majority of cases it is the seat of the magnetomotive forces which produce that flux of induction.

2. *The Armature*, the chief part of the dynamo which supports

[1] O. Frölich, *Elektrotechn. Zeitschrift*. vol. 14, p. 403, 1893.

the current conductors,[1] in which the electromotive forces are induced which cause the production of the current.

3. *The Air-gap*, which allows the passage of the flux of induction between the two chief ferromagnetic parts of the machine when moving relatively to each other.

These three parts may be recognised in all dynamos, whether used for producing direct current, ordinary two-phase or polyphase alternating current;[2] in discriminating these parts, their function is to be exclusively considered, and not their geometrical or mechanical arrangements. In almost all direct-current machines, the field-magnet is fixed, while the armature rotates. In many alternating current machines this is also the case, while in many others it is just the reverse, experience showing that a fixed armature has many advantages in practice.

As to the question which part is to be considered the field-magnet and which the armature, the rotation of the parts does not at all come into consideration. In every case the field-magnet is that part with reference to which the flux of induction remains unchanged, independently of any rotation. The magnetisation of the armature, on the contrary, undergoes periodic changes of its value, direction, or distribution, which cause the induction of electromotive forces. We will now consider more especially the arrangement of the chief parts of the magnetic circuit.

§ 141. **Arrangement of the Field-magnets (Framework).**—The most important consideration in designing the field-magnet arises from the necessity of making the reluctance of its magnetic circuit as small as possible. To this corresponds a minimum value of the magnetomotive force necessary for producing a given flux of induction—that is, as small a number of ampere-turns as possible. The unavoidable waste of energy by the heat developed in these latter is thereby reduced to a minimum. Accordingly, the field-magnets should be as short and of as large a section as possible, and constructed of iron of

[1] These usually consist of copper; compare, however, § 144.

[2] Direct current armatures do not materially differ, as regards magnetisation, from those for alternating currents. The difference is almost exclusively in the mode of winding and commuting.

the highest permeability. The constructive requirements affect, therefore, the form of the frame of the magnet; and, secondly, the material of which it is constructed.

As regards the shape of the framework, the section of the limbs should be circular, for then its perimeter is a minimum for a given section. In that case, the length of wire needed for magnetising a given section with a prescribed number of ampere-turns is as small as possible, as is also the loss of energy by the heat produced by the current. Limbs of circular section also are most easily turned and wound. Nevertheless, oval or rectangular sections often occur; in the latter, the edges, for several reasons, are best rounded.

In accordance with the principle, universally recognised at present, not to lengthen the magnetic circuit unnecessarily, the frames of the magnets of all well-constructed modern machines show compact forms. Besides the lessening of the magnetic reluctance, there is the advantage that the leakage is diminished as much as possible. The allowance for this latter circumstance influences the design of a framework in many other details, for which it is difficult to give general rules. It need only be mentioned that the general lines of the frame should follow as closely as possible the course of the lines of induction. In accordance with this, sharp bends in the magnetic circuit are to be avoided, as these cannot be followed by the lines of induction.

Further, two points of the circuit whose magnetic potential is markedly different, and between which, accordingly, there exists a considerable magnetomotive force, must not come too close anywhere, lest induction lines should pass between them, through the intervening air, and so be lost. In those parts, too, where there are considerable magnetomotive forces, no ferromagnetic bodies must be placed which serve for other purposes—such as base-plate, axes, bolts, &c.—for these may form injurious magnetic short circuits.[1] Finally, all projecting points, corners, edges, must be rounded off as far as possible, for experience shows that they always produce a certain leakage.

§ 142. **Continuation. Pole-pieces. Material.**—The shape of the pole-pieces which adapt themselves to the armature requires

[1] Compare the statement about short circuit through a base-plate (§ 130).

particular care. They must allow the induction lines to pass as uniformly as possible over the whole extent of the air-gap. They thereby directly influence the distribution of potential around the commutator, which must not deviate too much from the ideal sine curve.[1] A direct passage of induction lines between the neighbouring horns of the pole-pieces, instead of through the armature, must also be avoided.

The economical reasons for desiring to lessen the size of the field-magnet—or, rather, of the 'field ampere-turns'—find their natural counterpoise in the requirement that they must always dominate the armature ampere-turns, so that reaction of the armature and sparking do not increase too much (§ 134). This holds for the current-generator as well as for the electromotor. For the latter, lighter field-magnets are sometimes constructed, but only in cases in which it is important that they should weigh as little as possible.

In order to satisfy the requirement of as high permeability as possible, the best material for the field-magnets is the softest annealed wrought iron. As, however, the conditions previously discussed often lead to very complicated forms for the frame, which cannot well be forged, it is frequently preferable to cast it, by which, at the same time, joints are avoided (§ 144). Owing to the smaller permeability of cast iron, all sections must then be larger. Several kinds of steel, too, what is called 'mitis metal,'[2] or 'malleable cast,'[3] are used. In many cases wrought-iron cores for the limbs are used, with cast-iron casings and pole-pieces.

Owing to the almost complete constancy of the flux of induction through the field-magnets, the hysteresis of the material has not much influence. This has even a certain advantage in 'starting' the machine. Eddy currents are just

[1] Compare § 135; the self-induction of the armature coils is there disregarded. See also note, p. 278.

[2] Wrought iron made fusible by a slight addition of aluminium; this alloy is recommended by Silv. Thompson, *Dynamoelectrical Machinery*, 4th edition, p. 149, London, 1892.

[3] The author found (*Elektrotechn. Zeitschrift*, vol. **13**, p. 580, 1892) that this material has considerably higher permeability, and at the same time less hysteresis than ordinary cast iron. (Compare fig. 94, p. 349.)

[Quite recently several kinds of cast steel have been brought out, which almost equal wrought iron, though not the best Swedish specimens. H. d. B.]

as little to be feared, so that it is not necessary to split up the material, but it may be made solid.[1]

§ 143. **Arrangement of the Armature.** — The contrary of what has been stated in the last article holds for armatures, since in this case the magnetisation is, in every respect, an essentially variable quantity (§ 140). This part is therefore exclusively built up of thin stamped sheet-iron, or iron ribbon (in some cases also of round or, better, square iron wire), the thickness of which is a smaller or larger fraction of a millimetre, according to the quicker or slower variations of magnetisation (§ 187). In accordance with the object of this division, that of breaking the paths of parasitic eddy currents ('Foucault currents') in the body of the armature, it must be in planes at right angles to those paths—that is, parallel to the lines of induction, as well as to the direction of the motion.

As the section of the iron is lessened by the division,[2] the magnetic reluctance undergoes an unavoidable increase. On the other hand, demagnetising actions scarcely occur, as would be the case if the induction lines were at right angles to the planes of division (§ 30 E), instead of parallel to them. Taking that circumstance into account, the section of the armature should be such that its magnetisation is never near saturation, but rather remains on the steep ascending part of the curve of magnetisation. Its highest allowable value is then, for good wrought iron, about $\mathfrak{I} = 1200$ C.G.S. units, corresponding to an induction $\mathfrak{B} = 15{,}000$ C.G.S., as stated before (§ 126).

As to material, the purest possible soft wrought iron is always used, the best being Swedish, which combines high permeability with small hysteresis. The plate, after being stamped, is carefully annealed. As it is never possible entirely to suppress the heating due to hysteresis and eddy currents, to which must be added the heat due to the current itself, care must be

[1] For further details as to the construction of field-magnets, reference must be made to Kittler, *Handbuch der Elektrotechnik*, 2nd edition, vol. 1, chapter ix.

[2] The volume of the iron in an armature thus divided is, in percentages of the whole, with :—

Iron plate or strip . . .	about	80–90 per cent.
Square iron-wire . . .	„	70–80 „
Round „ „ . . .	„	60–70 „

taken to allow the heat to escape. This is not difficult with rapid rotation, since every armature naturally acts as a centrifugal ventilator. Good insulation of the separate iron discs is not of much importance; thin paper, or a coating of shellac or varnish, is used.[1] The armature core thus constructed is pressed together to form a compact mass by means of strong bolts, &c., and is then turned and wound. Hoop iron is almost exclusively used for flat ring armatures. We cannot here enter further upon the details of the construction and winding of the other chief forms of armatures (drum, ring, disc, toothed-ring, and pierced-disc armatures) made from sheet iron, and of the most varied patterns.[2]

§ 144. **Arrangement of the Interspace.**—The interspace always gives rise, as we have seen (§ 131), to the chief part of the magnetic resistance. As it is desirable to lessen this as much as possible, for economical reasons, the measures to be taken with this object must first of all be applied to the interspace. Useless air-spaces, especially joints between the separate parts, are, in the first case, to be avoided as much as possible, for they always produce magnetic reluctance and, moreover, leakage. In case joints cannot be avoided in the construction, the two faces, if possible, should be ground against each other, and the corresponding parts fastened tightly together by strong bolts (§§ 151, 152).

As regards the useful air-space, its width should be lessened as far as possible. With armatures which are completely wound, the width is prescribed by the section of the copper and the insulation, as well as by the space necessary for the free play of the armatures rapidly revolving between the pole-pieces.[3] In consequence of this, a lower limit is defined for the width of clearance.

[1] The iron scale produced in the annealing is sufficient for this purpose; it is, however, better to remove it, as it has been observed that such armatures often show remarkably high hysteresis, probably owing to the formation of Fe_3O_4, which is identical with the highly hysterestic magnetic iron ore.

[2] Silv. Thompson, *loc. cit.* chapter xiii; Kittler, *loc. cit.* chapter vii.

[3] Disturbances in working often occur owing to the irregular motion of the armature coils against the pole-pieces; an armature entirely covered by the winding can, of course, not be so accurately centred as one whose perimeter is entirely or partially formed by the iron core itself, which can, in being turned, be accurately centred, as will be seen in the types to be afterwards mentioned.

In imitation of the original Pacinotti ring, what are called *toothed-ring* armatures are often used. Their section is like that of a toothed wheel; the coiling is in the notches, while the magnetic reluctance is considerably decreased by the projecting teeth. More recently, *perforated armatures* have been introduced in which the conductor lies in a series of holes parallel to the axis of rotation, and near the perimeter of the armature. In this arrangement the air-gap is practically restricted to those cavities.

The obvious idea of using iron for the conductors instead of copper, has been realised, by which the air-gap, and therewith the resistance, is reduced to the smallest degree. The considerably higher electrical resistance of iron is, it is true, an objection. Although such iron coiling has hitherto been but little used, it is interesting from the purely magnetic point of view. We have mentioned it in the theoretical treatment of a ferromagnetic substance through which a current passes (§ 60). The construction of Forbes may be mentioned, which, indeed, represents the simplest plan of a dynamo. A solid cylinder or disc of wrought iron rotates with as little clearance as possible within a thick iron jacket, which completely encloses it. The field is produced by a few peripheral windings, which are firmly bedded in the iron jacket. Radial electromotive forces are induced in the rotating iron core, producing currents in the external circuit attached by sliding contacts to the axis and periphery.[1] We may, in conclusion, mention Fritsche's recently-constructed [2] dynamo, in which the conductors of the armature are also of soft iron.

§ 145. **Machines with Multiple Magnetic Circuit.**—In our theoretical developments (§§ 125–135) we have, for the sake of a readier survey, tacitly confined ourselves to those machines in which the useful flow of induction enters one part of the armature, and emerges at the opposite side, without closing further branch circuits through the frame of the magnet. Such machines are said to have a *single magnetic circuit*. The

[1] Forbes calls his machine a 'non-polar one'; it belongs to the type of machines without commutator, whose armatures continually cut the lines of induction, and the very earliest representative of which is Faraday's disc. They are still frequently called by the unsuitable name unipolar machines.

[2] W. Fritsche, *Die Gleichstrom-Dynamomaschine*, Berlin, 1892.

consideration is thus simplified as much as possible, as seen in the case of the Edison-Hopkinson dynamo. It has now, indeed, been shown by Rowland that theoretically it is better if the magnetic circuit of a dynamo is simple than if it is a multiple—that is, a branched—circuit. Nevertheless, theoretical reasons are not always the only ones to be considered in such cases.

As regards ordinary direct-current machines, with a divided magnetic circuit, the magnetic field is symmetrical. It is, further, easier to cope with the constructive difficulties which arise from the very considerable magnetic tractive forces. A continuous-current machine with simple magnetic circuit, if deprived of its commutator, constitutes a two-phase alternating current, the *frequency*[1] of which is equal to the number of turns. But as the latter cannot be increased so that the former is sufficiently high, it is necessary, in alternating machines, to increase the number of periods n times by arranging n magnetic circuits, and it is immaterial whether the armature or the field-magnets are movable.

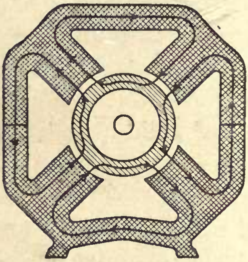

Fig. 34

Fig. 34 gives the plan of a machine with a quadruple magnetic circuit. The direction of the lines of induction is indicated by the arrows. In planning this machine, each of the four magnetic divided circuits must be designed on the principles which have been developed in the preceding articles. The product $\mathfrak{G} X$ of the prescribed flux of induction $\mathfrak{G}$ into the calculated magnetic reluctance X, which is to be kept as low as possible, gives the requisite magnetomotive force M which is to be assigned to the partial circuit in question. If that product is divided by $0{\cdot}4\,\pi$ (or multiplied by $0{\cdot}8$), the necessary number of ampere-turns is obtained.

§ 146. **Diagrams of various Magnetic Circuits.**—In fig. 35, p. 221, the magnetic circuits of fifteen types of machines,

[1] That is the number of complete periods in a second (equal to half the number of alternations), which, with the usual alternating currents now in use, is between 40 and 120; at the same time there is a decided tendency to diminish the number to the lower value.

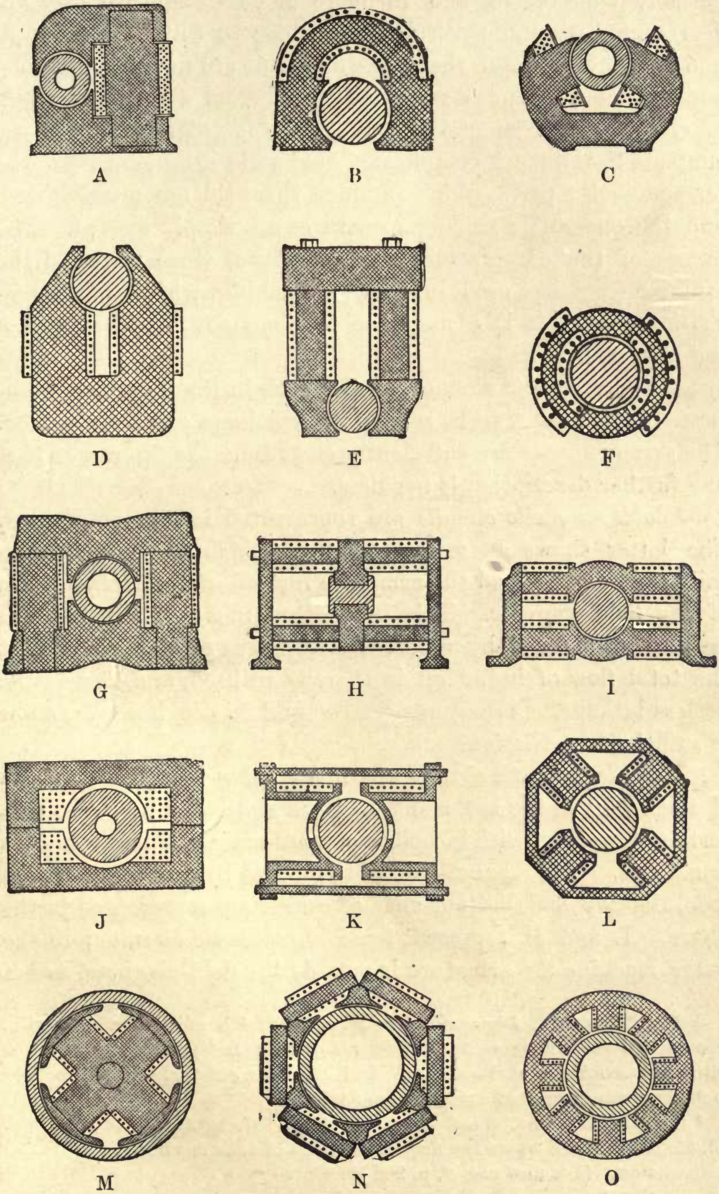

FIG. 35

actually constructed and in more or less extensive use, are represented diagrammatically, according to Silv. Thompson;[1] and they are so chosen that those diagrams are not given against which serious objections may be raised from a magnetic point of view. The order of the figures A–O is progressive from the simplest to the most complicated magnetic arrangement. The ferromagnetic parts which produce the field are cross-shaded, and the current-producing armatures are simply shaded. The names of the constructors are not given, since many of the arrangements are used in various machines with slight modifications, and this is a matter of less moment for the objects of the present book.

Single magnetic circuits are met with in the machines represented in figures A to E, several of which are extensively used. The arrangements are sufficiently clear from the figures, so that any further description is needless.

Double magnetic circuits are represented in figures F to J. The latter shows an arrangement in which the magnetising coils are wound round the armature instead of round the limbs of the field-magnets. This interesting mode of winding has been recommended by different electricians on the ground that the total flow of induction is thereby utilised, and there is no useless leakage. Armatures thus wound do not, however, allow of sufficient ventilation.

Multiple magnetic circuits are, finally, shown in the diagrams K to O. In K the last-mentioned principle of winding is, in a certain sense, met with in hollow cores of magnets[2] and spherical armatures. The magnetic circuit is closed by a number of iron rods, the top and bottom ones of which are represented in the figure. L and M represent fourfold magnetic circuits; in the latter machine the armature is outside the field-magnets, as has

[1] Silv. Thompson, *Dynamoelectric Machinery*, 4th edition, chapter viii., from which the substance and the figures of these last paragraphs are taken with kind permission of the author. In Kittler, *loc. cit.* figs. 462–466, over 60 such diagrams of magnetic circuits are depicted.

[2] Compare Grotrian, *Wied. Ann.* vol. **50**, p. 737, 1893, and du Bois, *ibid.* vol. **51**, p. 536, 1894, where the magnetic circuit of such machines is subjected to discussion. (See also note 2, p. 263.)

recently been often arranged (§ 140). N, finally, represents a sixfold, and O an eightfold, magnetic circuit.[1]

[1] The classification of machines according to the number of magnetic circuits is perhaps the most rational from the magnetic point of view; in practice, indeed, we still frequently speak of non-polar, unipolar (compare note, p. 219), bipolar, multipolar, as well as external polar, internal polar, consecutive polar machines, and the like.

CHAPTER IX

MAGNETIC CIRCUIT OF VARIOUS KINDS OF ELECTROMAGNETS AND TRANSFORMERS

A. *Physical Principles*

§ 147. **Magnetic Cycles.**—In the present chapter we shall continue to treat practical examples of the applications of the principles which hold for magnetic circuits; but from the great variety of arrangements we shall only select those which are typical, and have special theoretical interest. We have, however, previously to discuss the results of experimental investigations and theoretical considerations which have come into account, and which refer more particularly to the alteration of magnetic states. We shall first consider the phenomenon of magnetic hysteresis. In § 8 we have defined its general character, but in our subsequent developments we have always disregarded hysteresic processes. We shall limit ourselves, at present, to stating the main features of this important phenomenon, referring for experimental details to works in which ferromagnetic induction is more completely dealt with than is consistent with the object of the present book.[1]

We will, after Warburg,[2] subject a ferromagnetic substance of endless shape to a magnetic *cycle*, whereby we cause the magnetic intensity to pass through all values from $-\mathfrak{H}_G$ to $+\mathfrak{H}_G$, and back again to $-\mathfrak{H}_G$, where $\mathfrak{H}_G$ may have any chosen limiting value. Hysteresis then asserts itself by the fact that the corresponding successive values of magnetisation do not lie in a single curve, but, with a sufficient number of repetitions of the process, on a perfectly definite loop. For instance, the

[1] See Ewing, *Magn. Induction, &c.* chap. v.

[2] E. Warburg, *Wied. Ann.* vol. **13**, p. 141, 1881; Warburg and Hõnig, *Wied. Ann.* vol. **20**, p. 814, 1883. See also E. Cohn, *Wied. Ann.* vol. **6**, p. 388, 1879.

curve, fig. 36, *A*, represents such a typical *hysteresis loop*, which corresponds to the behaviour of a specimen of annealed steel

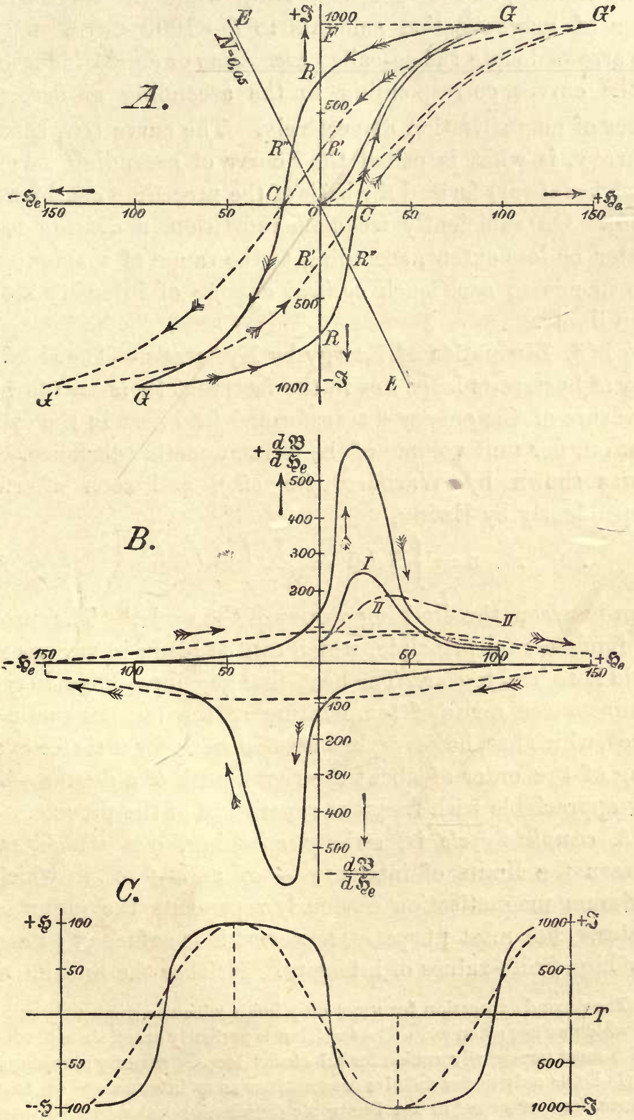

FIG. 36.—ANNEALED STEEL PIANOFORTE WIRE, $\mathfrak{H}_c$ = 20 C.G.S. units; $\mathfrak{u}$ = 84,000 ergs per cubic cm.

pianoforte wire. The numerical values are given in round numbers for the sake of clearness. The range of abscissæ is restricted to the values $(-100 < \mathfrak{H} < +100)$, while the corresponding range of magnetisation amounts to $(-1000 < \mathfrak{J} < +1000)$. The arrows denote the so-called *increasing* or *decreasing* branches of the curves corresponding to the ascending or descending values of magnetisation respectively. The curve $\overline{OG}$, that with no arrow, is what is called the 'curve of ascending reversals,' which has always formed the basis of the preceding considerations (§ 85). On sufficiently frequent repetition, a definite loop of greater or less extent, according to the range of magnetisation, corresponds to each such *cyclical* change of intensity between given limits.

§ 148. **Dissipation of Energy by Hysteresis.**—The chief property of hysteresis loops lies in the fact, that their area furnishes a measure of the energy $\mathfrak{u}$ transformed into heat in the cycle in question, per unit volume of the ferromagnetic substance. For, as was shown by Warburg (*loc. cit.*), and soon afterwards independently by Ewing,

$$(1) \quad . \quad . \quad \mathfrak{u} = \int \mathfrak{J}\, d\mathfrak{H} = \frac{1}{4\pi}\int \mathfrak{B}\, d\mathfrak{H}$$

taken between the limiting values of the cycle.[1] The proof of this fundamental principle, which can be given in various ways, would here lead too far. The heat disengaged produces, in certain circumstances, a rise of temperature of the ferromagnetic substance, which has, however, but a small value for a single cycle—being of the order of about the thousandth of a degree—but is very appreciable with frequent repetitions of the process.

A *complete cycle* is, strictly speaking, one which ranges between the limits of intensity $-\infty$ and $+\infty$, in which the maximum magnetisation obviously represents the range of ordinates. For most purposes, however, it is sufficient to keep in view high finite values of intensity. Neither the limiting value

[1] The second expression for $\mathfrak{u}$ not only holds with the same approximation with which we can often write $\mathfrak{B} = 4\pi\mathfrak{J}$, but is perfectly exact, since the integral $\int \mathfrak{H}\, d\mathfrak{H}$ must necessarily vanish for all closed loops. Strictly speaking, equation (I) holds only under definite assumptions as to interchange of heat; for instance, for isothermal or for isentropic (adiabatic) cycles (see Warburg and Hönig, *loc. cit.*, p. 817; Ewing, *Proc. Roy. Soc.*, vol. **23**, p. 22, 1881, and vol. **24**, p. 39, 1882). For most purposes, however, this is of small importance.

of magnetisation, nor the quantity of energy dissipated, can then greatly increase, and it is therefore sufficient to give the value of the latter for such approximately complete cycles. A great number of such determinations have been given by Ewing in his comprehensive investigation of this subject.[1] The value of $\mathfrak{u}$ varies from about 10,000 ergs per cubic centimetre for the softest annealed iron, up to over 200,000 ergs per cubic centimetre for glass-hard tungsten steel.

With incomplete cycles of restricted range of magnetisation, the dissipated energy exhibits a correspondingly smaller value. In his fundamental experiments Warburg had endeavoured to make clear its relation to the limiting value of magnetisation, but he found no definite law, and particularly no proportionality, between $\mathfrak{u}$ and $\mathfrak{J}^2_G$. Taking Ewing's data as basis, Steinmetz has recently found that the process may be represented approximately by the following empirical relation:

$$\mathfrak{u} = C\,(\mathfrak{J}_1 - \mathfrak{J}_2)^{1\cdot 6} \quad \text{or} \quad \mathfrak{u} = B\,(\mathfrak{B}_1 - \mathfrak{B}_2)^{1\cdot 6}$$

in which C and B are constants, depending on the nature of the material; $\mathfrak{J}_1$ and $\mathfrak{J}_2$ (or $\mathfrak{B}_1$ and $\mathfrak{B}_2$) represent the upper and lower limiting values, which need not be numerically equal. The difference $(\mathfrak{J}_1 - \mathfrak{J}_2)$—to be understood in the algebraical sense—represents the entire range of magnetisation, or the corresponding section of ordinates, which, moreover, need not be symmetrical to the axis of abscissæ; $\mathfrak{J}_1$ and $\mathfrak{J}_2$ may even have the same sign.[2] In special cases Ewing was able to represent the dissipation of energy as a linear function of the range of intensity.[3] W. Kunz[4] has recently thoroughly investigated the influence of temperature on hysteresis. The chief result he finds is that, on the whole, the dissipation of energy corresponding to a given range, decreases as the temperature increases, until at the temperature at which magnetisation almost vanishes, it would obviously altogether cease (compare note, p. 15). Shocks and vibrations, also, as

[1] Ewing, *Phil. Trans.*, vol. **176**, II., p. 523, 1885.

[2] Steinmetz, *Elektrotech. Zeitschrift*, vol. **12**, p. 62, 1891; vol. **13**, p. 519 *et seq.*, 1892.

[3] Ewing, *The Electrician*, vol. **28**, p. 635, 1892.

[4] W. Kunz, *Abhängigkeit der magn. Hyst. von der Temperatur*, Programm-Beilage Gymn. Darmstadt, Easter, 1893, and Dissertation, Tübingen, 1893.

well as rise of temperature, tend to counteract hysteresis, as already mentioned (§ 8).

The above discussions refer to the case in which the cycle is slowly performed, as is the case in the ordinary 'statical' method, by means of which the hysteresis loop may be determined (§ 207). The question whether, with rapid cycles, their duration would have an essential influence on the value of the dissipation of energy must still be regarded as an open one. It is connected with the phenomenon of magnetic time-lag, which has been but little investigated (note, p. 205). From the calorimetrical experiments of Warburg and Hönig, and of Tanakadaté, as well as from the investigations, by various methods, of J. and E. Hopkinson (§ 221), Evershed and Vignoles, Ayrton and Sumpner,[1] it must be considered probable that that influence is by no means an important one, at any rate, provided the duration of the cycle is not less than the one-hundredth of a second, which is about the lower limit for the period of alternating currents hitherto usual in practice (note, p. 220).

The quantity of energy dissipated per unit time in a magnetic cycle in consequence of hysteresis—that is, the power which must be practically regarded as lost[2]—is obviously obtained by multiplying the values of the hysteresis integral, or the area of the corresponding loop, into the volume of the ferromagnetic substance and into the number of cycles which are performed per unit time.

§ 149. **Influence of Shape. Retentivity. Coercive Intensity.**—The shape of the ferromagnetic substance exerts a considerable influence on the form of the hysteresis loop, as well as on the curve of magnetisation, which may be allowed for in the same manner by corresponding 'shearing' of the curve (§ 17). In fig. 36, *A*, p. 225, the full-line loop $\overline{CGCG}$ is supposed to be obtained with endless shapes, and therefore, in a certain sense, represents the *normal loop* for the material. Now

[1] Warburg and Hönig, *loc cit.*, Tanakadaté, *Phil. Mag.* [5], vol. **28**, p. 207, 1889; J. and B. Hopkinson, *Electrician*, vol. **29**, p. 510, 1892; Evershed and Vignoles, *ibid.*, vol **27**, p. 664, 1891; vol. **29**, pp. 583, 605, 1892; Ayrton and Sumpner, *ibid.*, vol. **29**, p. 615, 1892.

[2] The C.G.S. unit of power or activity is the erg per second; the practical unit is the watt or the kilowatt. Between these units we have the following relations:—1·34 British horse-power = 1 kilowatt = 1,000 watts = 10^{10} ergs per second.

let the straight line $\overline{EOE}$ correspond, for instance, to a demagnetising factor $N = 0{\cdot}05$ [1]; if the normal loop is sheared from it to the axis of ordinates, the dotted loop $\overline{CG'CG'}$ is obtained, which holds for the given value of N. If during a cycle, after magnetisation in a definite—say positive—direction, the intensity be allowed to diminish to the value 0, it is known that the magnetisation will not quite disappear, but will retain a certain positive value. This, the *residual magnetisation* $\mathfrak{J}_R$, is equal to the ordinate $\overline{OR}$ of the point of intersection R of the descending branch of the loop with the axis of ordinates. In the present example, $\mathfrak{J}_R = 700$ C.G.S. units. The ratio $\mathfrak{r}$ of the residual magnetisation to the previous limiting value is called the *retentivity*; this amounts, hence, to $\mathfrak{r} = \mathfrak{J}_R / \mathfrak{J}_G = 70$ per cent.

In order to reduce the magnetisation to the value 0, a certain intensity must be applied in the opposite direction, which is called the *coercive intensity*. This obviously corresponds to the abscissa $\overline{OC}$ of the intersection of the descending branch with the axis of abscissæ, and for the example given, amounts to $\mathfrak{H}_C = 20$ C.G.S. units. A glance at fig. 36, A, p. 225, shows that this latter quantity is not altered by the shearing—that is, it is independent of the shape. The case is different with the residual magnetisation, which is always reduced by the shearing, so that in the special case represented it only amounts to 300 C.G.S. units ($\mathfrak{r} = 30$ per cent.). This value, which corresponds to the ordinate $\overline{OR'}$, is also the ordinate of the intersection R'' of the line $\overline{OE}$ with the descending branch. The simple graphical determination of the retentivity of bodies of various shapes, which depends upon this, was given by Hopkinson,[2] who first applied the term 'coercive' in the manner indicated, to an unambiguous definite conception. Since both branches of the loop differ but little from a straight line, up to half the limiting value of the magnetisation, and are at the same time usually very steeply inclined to the axis of abscissæ compared with the straight line $[EOE]$—at any rate, when the value of N is not too small and the coercive intensity not too

[1] As would be the case, for instance, according to Table I., p. 41, for a piece of steel piano wire the length of which is about 25 times its diameter.

[2] Hopkinson, *Phil. Trans.*, vol. 176, p. 465, 1885.

great—an inspection of the triangle OCR'' for this case gives the convenient approximate equation:

$$(2) \quad . \quad . \quad . \quad . \qquad \mathfrak{I}_R = \frac{\mathfrak{H}_C}{N}$$

which, as observed, is only applicable for rather high values of N. In addition to $\mathfrak{I}_R$, the difference $\mathfrak{I}_G - \mathfrak{I}_R = \mathfrak{I}_E$ comes in many cases into account (§ 179); it is represented by the section of ordinates $\overline{FR}$ or $\overline{FR'}$, and may be defined as the *evanescent magnetisation*. This increases with the value of N. Both $\mathfrak{I}_R$ and $\mathfrak{H}_C$ increase with the range of magnetisation of the cycle, yet the unvarying coercive intensity may be regarded as a constant of the material for an (approximately) complete cycle. The area of the loop is no more altered by shearing than is the coercive intensity. Since the difference of abscissæ between the ascending and descending branches of the loop, as experience shows, is pretty constant for different values of the ordinates, that area, as a first approximation, is equal to four times the product of the coercive intensity into the limiting value of the magnetisation attained. Hence, after Hopkinson (*loc. cit.*), we may write, approximately,

$$(3) \quad . \quad . \quad . \qquad \mathfrak{u} = 4\,\mathfrak{I}_G\,\mathfrak{H}_C$$

Thus for a definite substance, the dissipation of energy for a given cycle only depends on the range of its magnetisation, and not on its shape. The percentage differences of the ordinates too, corresponding to the ascending or to the descending branch, are scarcely affected by it, as shown in fig. 36, *A*. The statement so frequently made, that hysteresis comes less into account in short bodies, applies only to the fact that in the first place, for a given limiting value of the intensity of the impressed field, the range of magnetisation, and with it the coercive intensity, are less, from what has previously been said, and in the second place to the considerable decrease of the retentivity which has been mentioned.

§ 150. **Permanent Magnets.**—Although the construction of powerful permanent magnets is now of less interest than at the time when the relations between electricity and magnetism had not been revealed, they are still important, especially for

measurements in physics. In most cases it is a question of the greatest constancy of the residual magnetism, so that they can be really called *permanent*; in others it depends more on a high value of magnetisation; we shall see that these two conditions, in a certain way, are inconsistent with each other. From the most recent experiments it appears as if we had come nearer to the possibility of preparing magnets sufficiently permanent, which had long been doubted. The conditions to be complied with concern the shape, the material, and the treatment of the magnets. It must be premised that an unclosed permanent magnet, left to itself, is in a state of unstable equilibrium, as the magnetism must maintain itself, notwithstanding its own self-demagnetising intensity. In the long run, unavoidable shocks, vibrations, and rises of temperature act in opposition to that condition (§ 148 the action of the latter is not always entirely got rid of by subsequent coolings); and, *ceteris paribus*, these will occur in so much greater measure the greater the demagnetising action—that is, the greater are, firstly, the magnetisation itself, and, secondly, the demagnetising factor N.

It appears thus to need no further explanation that, in the first place, that shape must be chosen which has the least demagnetising tendency; accordingly, short compact forms must not be used, but elongated ones. If the magnet is not to be used for actions at a distance, it is better to bend it into a circuit, with as narrow an air space as possible; on this depends the well-known advantages of the 'horseshoe' form, as well as the rule that such an one should always be connected by the keeper, or that bar magnets be kept in pairs between two keepers. We shall subsequently (§ 197) describe a 'permanent field standard' which is constructed in conformity with the above principles. Jamin's 'lamellar magnets' have met with extended application; they are made of a number of steel strips, of such dimensions that the central one slightly projects, while the others are stepped back; this, as experiment shows, is less favourable to mutual demagnetisation.

As regards the material, if the chief object is to obtain high values of permanent magnetisation, a kind of steel must be chosen the hysteresis loop of which, after shearing according to the shape chosen, gives as high retentivity as possible (§ 149).

If, on the contrary, constancy is the chief object, it follows from the above that, for the sake of less demagnetisation, it is better to be satisfied with a weaker magnet; this assertion is also confirmed by experience. In this case high coercive intensity is of greater importance than retentivity,[1] the former appearing, in a certain sense, to bring with it increased stability of magnetisation.

A permanent magnet, as we have already said, is very sensitive to changes of temperature as well as to vibrations, shocks, &c. These deleterious influences must therefore as much as possible be avoided in use, as well as, of course, direct magnetic or electrical influence. It has been shown mainly by the researches of Strouhal and Barus that that sensitiveness may in a certain sense be diminished by previously 'ageing' the magnet. For this purpose it is accustomed for some time to an exaggerated rough treatment, consisting of boilings, shocks, blows, being allowed to fall, repeated magnetisation, and the like; this treatment may be called *artificial ageing*; magnets thus treated usually keep better. In the choice of material regard is had to the chemical composition, structure, texture, and homogeneity; it is usual to be guided by the points of view mentioned above, though we must finally be guided mainly by experiment; but in addition to this the following factors are important. Firstly, the previous treatment in the fire in forging, then the temperature and other details in hardening and tempering the steel, as well as, finally, the mode of magnetisation, which at present is exclusively effected by coils, and no longer by stroking with other magnets.[2]

§ 151. **Magnetic Reluctance in Joints.**—We have repeatedly called attention (§§ 16, 107, 144) to the influence which even

[1] These two properties are by no means correlated—at any rate, not with closed magnetic circuits; in this case with very soft iron we often have $\mathfrak{k} \geqq 90$ per cent., and $\mathfrak{H}_c \geqq 1$ C.G.S. unit. With steel $\mathfrak{k}$ is always less; on the contrary, $\mathfrak{H}_c$ is far greater than the values given above, while, *e.g.*, with hard tungsten steel the coercive intensity in certain circumstances amounts to more than 50 C.G.S. units.

[2] Older statements as to the production of permanent magnets are met with in Lamont, *Handbuch des Magnetismus*, 1867. See further Jamin, *Compt. Rend*, vol. **76**, p. 1153, 1872, and vol. **77**, p. 305, 1873; Strouhal and Barus, *Wied. Ann.*, vol. **11**, p. 930, 1880, vol. **20**, pp. 525, 621, 662, 1883; Holborn, *Zeitschrift für Instrumenten-Kunde*, vol. **11**, p. 113, 1891, as well as many

the finest joints and cracks exert in magnetic circuits, and which shows itself by actions not confined to the joints only—that is, by demagnetisation and leakage. We shall here go more minutely into the recent literature of this subject. J. J. Thomson and H. F. Newall first showed[1] that transverse joints in iron bars exert a considerable demagnetising action and leakage. They first determined the magnetisation in a given field: after cutting through the bars and putting them together again a surprising decrease of magnetisation was observed, which was only partially removed by grinding the ends; by interposing an increasing number of thin indifferent layers, the decrease of course always became greater. They also determined the leakage by means of iron filings (§§ 4, 189), and then depicted the magnetic figures obtained (*loc. cit.*)

Ewing and Low[2] then worked at this subject; they first investigated an iron bar (12·7 cm. long by 0·49 sq. cm. cross-section) by means of the ballistic bar-and-yoke method (§ 218). After the curve of magnetisation of the undivided bar had been determined, the bar was cut through transversely, and the ends fitted as usual by careful scraping and testing on a plane surface. Both halves were then put together again in as close contact as possible, and the curve of magnetisation again determined. It was found sheared in respect of the first (fig. 37). It follows from this that a joint acts like an air-gap, although there must undoubtedly be an actual contact of the ends of the iron rod at many points of the surface of separation. The demagnetising action of the joints is best expressed by stating the width d of the *equivalent air-gap*, between two geometrically plane faces, which would produce the same demagnetisation—that is, would offer the same magnetic reluctance, whereby the

other data. The most recent collation of the literature, which for the most part is widely scattered, as well as of tables of constants and so forth, is given by Silv. Thompson, the *Electromagnet*, 2nd edition (3rd edition of his Cantor Lectures), chap. xvi., London, 1892.

[1] J. J. Thomson and H. F. Newall, *Proc. Phil. Soc.*, Cambridge, vol. **6**, II., p. 84, 1887.

[2] Ewing and Low, *Phil. Mag.* [5], vol. **26**, p. 274, 1888; Ewing, *ibid.*, vol. **34**, p. 820, 1892.

[In a recent paper by Houston and Kennelly in the *Electrician*, vol. **35**, p. 160, 1895, the subject is dealt with on much the same lines as in the text, and a 'factor of safety' (against demagnetisation) introduced. H. du B.]

former may be deduced in the usual way from the difference of abscissæ $\Delta \mathfrak{H}$ of the two curves of magnetisation.

For the purpose of this calculation we will consider the bar in its closed yoke (fig. 38, p. 235); in first approximation as a toroid, slit radially, the perimeter of which is $2\pi r_1 = L$ (the length of the bar). In so narrow a slit the simple equation (VII) § 82 may be used

$$(4) \quad . \quad . \quad . \qquad N = \frac{2d}{r_1} = \frac{4\pi d}{L}$$

putting this value into the equation $\Delta \mathfrak{H} = N \mathfrak{J}$, we find

$$(5) \quad . \quad . \quad . \qquad d = \frac{L}{4\pi \mathfrak{J}} \Delta \mathfrak{H}$$

By means of the last equation Ewing and Low calculated the width of the equivalent air-gap, which for two iron bars was found equal to about 0·03 mm. This value was found to be

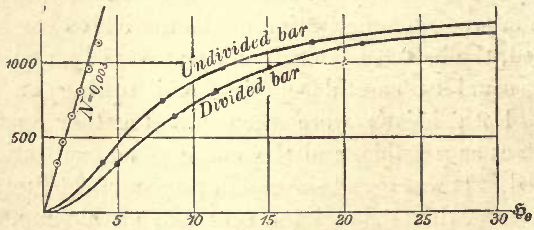

FIG. 37

fairly independent of the magnetisation, as is shown by the almost straight line in fig. 37, which represents a value $N = 0{\cdot}003$. It is difficult to decide whether that equivalent air-gap did, in fact, represent the mean distance of the faces, although such a considerable value of the latter is not probable. It can, with certainty, be alleged that a surface produced in the manner described differs materially from a highly polished plane mirror, and still more from a geometrical plane. Ewing and Low have further confirmed that the interposition of a single gold foil had not any further appreciable influence on the magnetic reluctance. The thickness of such a foil amounts, as we know, to only a fraction of the wave-length of sodium light—that is, it is of the same order of magnitude as that by which a good metal mirror differs from an absolute geometrical plane.

§ 152. **Influence of Applied Longitudinal Pressure.**—The same experimenters then investigated the influence of longitudinal pressure on the magnetic behaviour of a joint between two halves of a bar by the apparatus represented in fig. 38. At the same time they examined how far the magnetic properties of the material itself were affected by pressure (§ 107).

They found that the demagnetising action—or, in other words, the magnetic reluctance—of a joint arranged as above decreases with increase of pressure, and that for a pressure of over 200 kg.-weight per sq. cm. no difference could be detected between a divided and an undivided bar. In this case, as was to be expected, the interposition of a gold foil had a small,

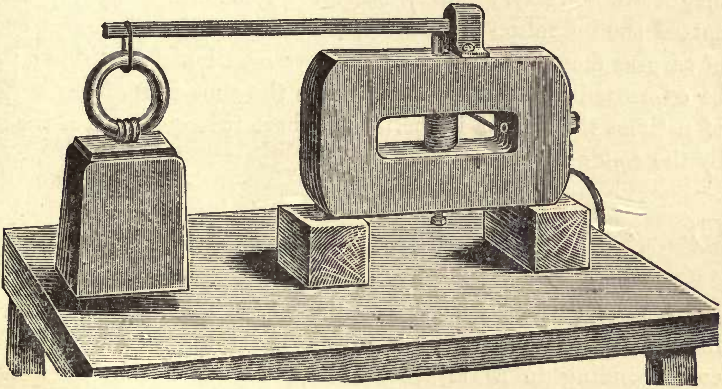

FIG. 38

but appreciable, influence. It is here to be observed that the magnetic pull (§ 102) was less than 1 kg.-weight per sq. cm., so that it need scarcely be considered of any influence in comparison with the above value of the external pressure.

Ewing and Low further made experiments with rough joints—that is to say, such as were bounded by surfaces simply turned and not further fitted. It was found that the width of the equivalent air-gap for such rough joints was about 0·05 mm., and was only reduced to 0·04 mm. by a pressure which would have entirely nullified the magnetic reluctance of a joint between properly prepared faces. Experiments were made with bars which had not one transverse joint only, but three to seven joints.

The results given offer considerable practical interest. They lead to the rule that the less a magnetic circuit is to differ from a closed one, the more carefully must the detrimental magnetic reluctance of superfluous joints be avoided; that, moreover, where, for constructive reasons, joints cannot be avoided, the surfaces must be carefully fitted and pressed against each other with great force. In magnetic circuits with a wide air-gap an additional slit of $\frac{1}{30}$ mm. equivalent width is not of much moment, so that in such cases the joints have scarcely any influence.[1]

§ 153. **Time-variations of the Magnetic Conditions.**—We have hitherto limited ourselves to the consideration of invariable lasting magnetic conditions—that is, to the case of *stationary* magnetisation. We have, however, incidentally (§ 64) discussed the induction of electromotive forces E in consequence of *varying* magnetisation; it was there mentioned that this may be expressed in absolute measure, as the time-rate of variation of n-times the flux of induction $\mathfrak{G}$ where it is encircled n-times by the conductor; that is,

$$(6) \quad . \quad . \quad . \quad E = \frac{d\,(n\,\mathfrak{G})}{d\,T}$$

We have then, further, exclusively considered the time-integral of E—or, in other words, the total current impulse corresponding to a variation $\delta\,\mathfrak{G}$—quite apart from its time-variation, and which affords the most suitable method of measuring it.

We shall now investigate more minutely variations of this kind, and we shall again elucidate the somewhat complicated phenomena which occur by the example of a toroid, either closed or divided radially, wound uniformly with n turns (resistance R, radius of the toroid r_1, perimeter $2\,\pi\,r_1 = L$, cross-section S). At the beginning of the time ($T = 0$) let a constant external electromotive force E_e act suddenly on such an 'induction coil'; this would correspond to the steady current $I_e = E_e / R$, to the immediate establishment of which there is, however, an impediment in the form of a self-induced counter-

[1] Compare an investigation by Czermak and Hausmaninger, *Wiener Berichte*, vol. **98**, 2 Abth., p. 1142, 1889.

electromotive force E_i.[1] The corresponding differential equation is

$$(7) \qquad IR = E_e - E_i = E_e - \frac{d(n\mathfrak{G})}{dT} = E_e - \frac{d(n\mathfrak{G})}{dI}\frac{dI}{dT}$$

T being the time, and I the actual current.

Besides the constants E and R, and the single variable I, the current flowing in each instant, the derived function or differential coefficient $d(n\mathfrak{G})/dI$ occurs, which plays an important part in what follows. It is quite generally called the *coefficient of self-induction*, or the *self-inductance*, Λ of the coil, independently of the simple arrangement assumed in the present example. From the well-known equations

$$\mathfrak{H}_e = \frac{4\pi n I}{L} \quad \text{and} \quad \mathfrak{G} = S\mathfrak{B}$$

we obtain, in accordance with the above definition,

$$(8) \qquad \Lambda = \frac{d(n\mathfrak{G})}{dI} = \frac{4\pi n^2 S}{L}\frac{d\mathfrak{B}}{d\mathfrak{H}_e} = \frac{2n^2 S}{r_1}\frac{d\mathfrak{B}}{d\mathfrak{H}_e}$$

The variable coefficient Λ depends then, in the first place, on the geometrical configuration of the coil, and has the dimension S/L—that is, a length.[2] It is, further, proportional to the differential coefficient of the induction $\mathfrak{B}$ in the ferromagnetic core, with respect to the intensity $\mathfrak{H}_e$ of the field of the coil. We shall, therefore, at once specially consider this differential coefficient.

§ 154. **Discussion of the Function $d\mathfrak{B}/d\mathfrak{H}_e$.**—In the first place, we get from the fundamental equation [§ 11, equation (13)]

$$\mathfrak{B} = 4\pi\mathfrak{J} + \mathfrak{H}_e$$

[1] We shall here use suffixes similar to those in § 53. Parasitic (Foucault's) eddy currents in the ferromagnetic substance itself will be expressly disregarded in the sequel; for this purpose we may consider it as having infinitely small electrical conductivity, or as being divided so that it has an infinitely fine-grained texture. Further, the electrostatic capacity of the coil will be disregarded, and its resistance R will be considered constant—that is, the current will be assumed constant for the whole length of the conductor, and uniformly distributed throughout its cross section. In the extremely rapid alternations of current, to which the attention of experimenters is now being pre-eminently directed, these assumptions are inadmissible; but in the present case we are considering variations which are comparatively slow.

[2] The C.G.S. unit for self-inductance, therefore, is of course the centimetre; the official delegates to the International Congress of Electricians at Chicago in 1893 fixed the *henry* (10^9 cm.) as practical unit, a length which hitherto has also been termed the Secohm or Quadrant.

the further equation

$$(9) \qquad \frac{d\mathfrak{B}}{d\mathfrak{H}_e} = 4\pi \frac{d\mathfrak{J}}{d\mathfrak{H}_e} + 1$$

In so far as we may neglect the second term on the right of the former equation, in comparison with the first, we can do so with the corresponding unit in the second equation. From the value of $[d\mathfrak{B}_1/d\mathfrak{H}_e]$ or of $[d\mathfrak{J}_1/d\mathfrak{H}_e]$, for a given value of the induction $\mathfrak{B}_1$ or the magnetisation $\mathfrak{J}_1$ with a closed toroid,[1] we shall deduce that which, *ceteris paribus*, corresponds to a finite demagnetising factor N. Apart from the algebraical sign [§ 53, equation (1)], we have

$$\mathfrak{H}_e = \mathfrak{H}_t + \mathfrak{H}_i = [\mathfrak{H}_e] + N\mathfrak{J}$$

By differentiation with respect to $\mathfrak{J}$ we obtain

$$(10) \qquad \frac{d\mathfrak{H}_e}{d\mathfrak{J}} = \left[\frac{d\mathfrak{H}_e}{d\mathfrak{J}}\right] + N$$

which also results from the fact that by 'shearing,' the tangent of the angle of inclination of each element of the curve to the axis of ordinates is increased by an amount proportional to N.

If now the latter equation is put in (9), we have

$$(11) \qquad \frac{d\mathfrak{B}_1}{d\mathfrak{H}_e} = \frac{(4\pi + N)\left[\dfrac{d\mathfrak{J}_1}{d\mathfrak{H}_e}\right] + 1}{N\left[\dfrac{d\mathfrak{J}_1}{d\mathfrak{H}_e}\right] + 1}$$

The general graph of the function $[d\mathfrak{B}/d\mathfrak{H}_e]$, or of $d\mathfrak{B}/d\mathfrak{H}_e$, which, when there is hysteresis, is no longer a single-valued function, is represented in fig. 36, *B*, p. 225. These curves exactly correspond to the hysterestic cycle for annealed steel wire represented in *A* (§ 147), and, therefore, scarcely require any further explanation. The full-line curves for a closed toroid show, in the first place, a characteristic prominence, which is still more strongly marked with soft iron, and is therefore difficult to represent (*cf.* fig. 41, p. 243); and, in the second place, a discontinuity corresponding to *G*, the top of the hysteresis loop. The full-line curve

[1] The expressions referring to closed toroids, $N = 0$, are in the sequel put in square brackets. Knott has proposed the names 'differential susceptibility,' or respectively 'permeability,' for the differential coefficients $[d\mathfrak{J}/d\mathfrak{H}_e]$ and $[d\mathfrak{B}/d\mathfrak{H}_e]$, which we, however, shall not adopt here.

I in the upper quadrant on the right corresponds to the curve of increasing magnetisation $\overline{OG}$. The dot-and-dash curve II represents the permeability $\mu =$ funct. $(\mathfrak{H}_e)$. As is obvious, both have a similar course, but are not identical except in their point of intersection, which corresponds to the maximum permeability —that is, to that point on $\overline{OG}$ in which this curve is touched by a straight line through the origin of co-ordinates. There can be no question of a constant value for $[d\mathfrak{B}/d\mathfrak{H}_e]$, and therefore neither for Λ. With the dotted curves, which correspond to a divided toroid ($N = 0{\cdot}05$), and which shows the above-mentioned properties in a less pronounced degree, an approximately constant mean value may, indeed, be introduced within a certain range. This is so much the more justifiable the greater the value of the demagnetising factor; the nearer, therefore, $\mathfrak{B}$ is proportional to $\mathfrak{H}_e$. In this case, obviously, $d\mathfrak{B}/d\mathfrak{H}_e = \mathfrak{B}/\mathfrak{H}_e =$ constant. In a toroid of indifferent material (§ 9), evidently, $d\mathfrak{B}/d\mathfrak{H}_e = 1$. Hence, on this assumption, we have

$$(12) \quad . \quad \Lambda = \frac{4\pi n^2 S}{L} = \frac{2n^2 S}{r_1} = \text{constant}$$

Only in this case can the differential equation (7) of § 153 be directly integrated. It may, however, first be transformed as follows, even with variable self-induction:

$$(13) \quad . \quad \frac{dI}{dT} = \frac{E_e - IR}{\Lambda} = \frac{I_e - I}{\theta} \qquad [T = 0;\ I = 0]$$

in which $E_e/R = I_e$, the steady currrent, and $\Lambda/R = \theta$ is put equal to what is called the *time-ratio*, which, in the general case, is variable. The initial conditions are put in brackets.

§ 155. **Simplification with Constant Self-inductance.**—The integration of equation (13) is only directly possible when θ happens to be constant, and gives

$$(14) \quad . \quad . \quad . \quad I = I_e\,(1 - e^{-T/\theta})$$

This is von Helmholtz's logarithmic law of the gradual rise of a current in an induction coil.[1]

If, on the other hand, the impressed electromotive force is removed at the beginning of the time, and the coil is then suddenly

[1] Helmholtz, *Pogg. Ann.*, vol. **83**, p. 511, 1851. *Wiss. Abhandl.*, vol. **1**, p. 434, Leipzig, 1882

short-circuited—not interrupted—then the following analogous law is found for the gradual decay of the current (*loc. cit.* p. 537 or 459):

$$(15) \quad . \quad . \quad . \qquad I = I_e \, \varepsilon^{-T/\theta}$$

Accordingly, the time-ratio is in this case the time between the moment of short circuiting and that at which the current has decreased to $1/\varepsilon$ of its value, or, by equation (14), the time required to reach a fraction $(\varepsilon - 1)/\varepsilon$ of its steady value from the moment of making the circuit.

Of greater importance is the case in which the impressed electromotive force acting on the induction coil is sinusoidal—that is, is a sine-function of the time T, of period τ, or frequency $N = 1/\tau$—and is accordingly represented by the following equation:

$$E = E_M \sin (2 \pi N T)$$

It may be shown that the alternating current in the coil satisfies the following equation:

$$(16) \quad . \quad . \qquad I = \frac{E_M}{J} \sin 2 \pi \left(\frac{T}{\tau} - \chi\right)$$

in which, for shortness,

$$(17) \quad . \quad . \qquad \chi = \frac{1}{2 \pi} \tan^{-1} (2 \pi N \theta)$$

This current, therefore, is also represented by a sine function, the phase of which is retarded against that of the impressed electromotive force by the given fraction χ, as given by equation (17).[1] The letter J in the above equation (16) is an abbreviation for the expression

$$(18) \quad . \qquad J = \sqrt{R^2 + Y^2} = \sqrt{R^2 + 4 \pi^2 N^2 \Lambda^2}$$

In induction coils it is usual to call

R, the resistance according to Ohm's law, or the '*Ohmic resistance.*'

$Y = 2 \pi N \Lambda$, the 'inductive resistance.'

$J = \sqrt{R^2 + Y^2}$, the 'impedance.'

[1] In acoustical and optical discussions, differences of phase or of path are usually expressed as fractions of the period or of the wave-length respectively, as has been done in the text. In the literature of alternating currents, phases are frequently expressed as parts of the circumference, given in degrees; the above fractions would in that case have to be multiplied by 360.

By the latter the throttling action opposing the sinusoidal electromotive forces is expressed analogous to that which Ohm's resistance offers to steady differences of potential.[1] The relation between R, Y, and J may be represented in an instructive manner by a right-angled triangle (fig. 39), the acute angle of which, expressed as fraction of the circumference, is equal to the difference of phase. It is not, however, our object in this paragraph to discuss the simple processes with constant self-induction.[2] We turn, rather, to the influence which the introduction of ferromagnetic cores with variable or multiple values of $d\mathfrak{B}/d\mathfrak{H}_e$ has on the character of those phenomena, and which we can most clearly represent graphically. We shall in this, again, consider in the above order the first problem of the growth and decay of electric currents, as well as, further on, in § 157, the properties of alternating currents under the influence of sinusoidal electromotive forces.

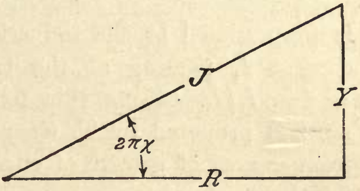

Fig. 39

§ 156. **Influence of Variable Self-induction.**—Let the curve $\overline{OP}$ in fig. 40, p. 242, be a given curve of rise of current $I =$ funct. (T). Its asymptote $\overline{M'M}$ then corresponds to the steady current I_e. It follows from the general differential equation (13) of § 154—which, as stated, holds also for variable self-induction—that

$$(19) \quad . \quad . \quad . \quad . \quad \theta = \frac{\partial T}{\partial I}(I_e - I)$$

The time-ratio θ for a point P is therefore equal to the tangent of the inclination of the curve to the axis of ordinates, multiplied into the portion $\overline{QP}$ of the ordinates, which represents the deficiency by which the actual current falls short of the final steady current. The integration of the latter curve up to the time T_1 (which corresponds to the point P of the curve) at

[1] The contrivance frequently used for this purpose in dealing with alternating currents is called a 'choking coil.'

[2] Compare, for instance, the elementary graphical representations in Fleming, *Alternate Current Transformer*, vol. 1, pp. 95–116, London, 1890.

once shows that the area of the shaded surface enclosed by the line $PQM'OP$, that is (see the fundamental equations of § 153),

$$(20)\qquad (PQM'O) = \int_0^{T_1} (I_e - I)\,dT = \frac{1}{R}\int_0^{I_1} A\,dI = \frac{nS}{R}\mathfrak{B}_1$$

is proportional to the induction $\mathfrak{B}_1$, which corresponds to the current I_1 passing at the time T_1, or to the intensity $\mathfrak{H}_e = 4\pi n I_1/L$. From this follows a theoretically interesting method proposed by T. Gray for deducing curves of induction from curves of growth of current, to which we shall revert in § 221.

The dotted (I, T)-curve $\overline{ON}$ (fig. 40) was obtained by the experimenter in question[1] with a large electromagnet. To this

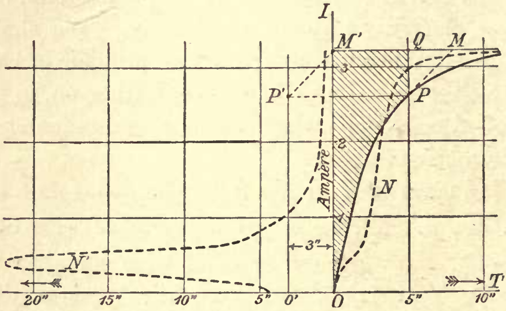

FIG. 40

corresponds in the left quadrant the dotted curve N', which represents the variable time-ratio $\theta = A/R$ (absc.) as function of the current (ord.) (see fig. 36, B, p. 225). For the sake of better comparison the full-line curve $\overline{OP}$ was calculated from Helmholtz's equation (14), taking as basis a constant time-ratio $\theta = 3''$, which is obviously represented by the line $O'P'$ parallel to the axis of ordinates. A glance at the figure shows

[1] T. Gray, *Phil. Trans.* vol. **184**, A, p. 531, 1893; the principal constants of the electromagnet were: $L = 265$ cm., $n = 3{,}840$, $R = 11{\cdot}5$ ohms; in the experiments here given the magnetic circuit was closed, the measurements on open circuit are not free from objection. It scarcely needs mention that the investigations of § 153, *et seq.*, though elucidated by the instance of the toroid, are not confined to this simple form. For shortness sake, an $(X\ Y)$-curve denotes in the sequel one which represents X as a function of Y.

the characteristic deviations which are due to the variable self-induction, especially the slower or quicker increase of the current, according as the variable time-ratio is greater or less than 3″ (compare § 170).

It is further to be observed that in either case the tangent $\overline{PM}$ in a point P of the (I, T)-curve runs parallel to the straight line, which connects the corresponding point P' of the (θ, I)-curve with the fixed point M'. Sumpner has based on this a graphical method by means of which the (I, T)-curve can be constructed for an electromagnet, the induction-curve and dimensions of which are known; and this, too, not only for

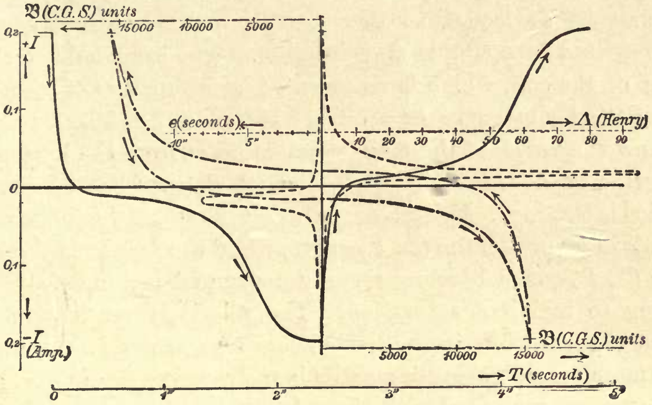

FIG. 41

the case of the simple rise and fall of the current, but also when the external electromotive force is one given by any arbitrary function of the time.[1]

We shall, in conclusion, reproduce in fig. 41, as example, one of the many diagrams given by Gray (*loc. cit.*). It embraces, as is seen, an entire cycle. In this the

(I, T)-curves are full; ————
(θ, I)- or (Λ, I)-curves are dashed; - - - - - - - -
$(\mathfrak{B}, I)$-curves are dot and dash lines. ·—·—·—

The scales for the various curves are drawn like them, only

[1] Sumpner, *Phil. Mag.* [5], vol. **25**, p. 470, Plate III., 1888. See also Fleming, *loc cit.*, vol. **1**, p. 263.

thinner to avoid confusion. On all the curves the points of the same height of the ordinates correspond. The figure needs, then, no further explanation. For further details we refer to the paper of T. Gray which has been quoted.

§ 157. **Sinusoidal Electromotive Forces.**—If the impressed electromotive force acting on the induction coil is a sinusoidal one, the form of the curve of the alternating current discussed in § 155 is very considerably modified by the introduction of a ferromagnetic core. Let us first consider the simpler case that the self-induction is extremely small, which, according to equation (12), would be effected by making the number of turns and the section as small as possible. Its influence, accordingly, will at first be neglected, so that the alternating current can be represented by a simple sine-function. The same holds for the field of the coil, which is represented as a function of time by the dotted sine-curve of arbitrary period in fig. 36, *C*, p. 225. If now the values of the magnetisation taken from the hysteresis diagram A above are plotted, the $(\mathfrak{I}, T)$-curve of the same period is obtained. This shows, as is seen, a retardation of its zero-points in respect of the $(\mathfrak{H}, T)$-curve, while the maxima coincide.[1] The $(\mathfrak{I}, T)$-curve becomes thereby unsymmetrical and flattened owing to incipient saturation. The relatively small displacement of zero-points in the special case represented is, obviously, the more considerable the smaller is the range of the intensity in the cycle as compared with the value of the coercive intensity.

Taking now a considerable value of self-induction, it may be theoretically proved, and partly also confirmed by experiment, that the (I, T)-curve is chiefly influenced by it in the following points:

As soon as the magnetisation begins to approach saturation, characteristic projections, standing out on the sine-curve, are shown on the (I, T)-curve, which, as the saturation increases, pass into sharp peaks. The choking action of the coil considerably diminishes after this.

Hysteresis occasions an asymmetry of the ascending and

[1] We can, strictly, only speak of a retardation of phase in the case of two sine-functions of equal period. It may also be remarked that retardation of the null point is in consequence of purely statical hysteresis, and is not necessarily connected with the question of the magnetic retardation.

descending branches of the (I, T)-curve, as well as in the retardation of zero-points of the $(\mathfrak{I}, T)$-curve in respect of them. Their altered form reacts in turn on the latter, which thereby differ from the shape previously discussed, and again approach the form of a sine-function. The periodical processes, moreover, which come into play in consequence of self-induction, show a certain tendency to revert to the simpler original form represented by the sine-curve, as the higher harmonic components are more choked than the fundamental periodic. An exhaustive discussion of this peculiar feature would lead too far.

There is, finally, in the ferromagnetic substance a continual dissipation of energy into heat, which may be calculated by the data in § 148. We shall revert to the phenomena previously discussed in dealing with the magnetic circuit of induction coils and transformers.

B. *Electromagnets for exerting different kinds of Pull*

§ 158. **Principle of Least Reluctance.**—Electromotors, in the narrower sense of the word, as discussed in the previous chapter, which on continuous rotation transform a large amount of energy, and may be considered as inversions of dynamo machines, have almost entirely driven from the field the arrangements previously devised for this purpose. There are, however, in the various applications of electromagnetism many forms of apparatus—for instance, measuring instruments, arc lamps, regulators, relays, bells, telephones, and many others—in which what is usually a very small transformation of energy is scarcely of any importance; the action of which is, however, to produce a relative change of position of their parts. We shall now consider such 'electromagnetic mechanism' from some general points of view. There can, of course, in this book be no attempt at even an approximately complete review of the special contrivances used in an infinite variety of particular cases.

Their action is ruled by the following principle, which has been more or less distinctly expressed by several authors,[1] and which in many practical cases serves as a useful guide, although it can scarcely claim any special theoretical importance :—

[1] Fleming, *loc. cit.*, vol. 1, pp. 28, 71. Silv. Thompson, *The Electromagnet*, 2nd edition, p. 277, 1892. Part of the figures and the contents of the previous

I. *The configuration of an electromagnetic system exhibits a tendency to vary in such a way that its reluctance reaches a minimum value.*

This condition is obviously tantamount to saying that, with a given magnetomotive force assumed, the flux of induction tends to a maximum. The latter principle, again, might be deduced, with certain restrictions, from a consideration of the general equations of electromagnetic energy. Considering the subordinate scientific importance, and the want of definiteness of the idea of magnetic reluctance (§ 119), we may dispense with any rigorous mathematical proof of the above principle. Its practical application will, in the sequel, be explained in reference to a number of examples.

§ 159. **Mechanisms depending on Electromagnetism,** in which, in conformity with the above principle, an open magnetic circuit tends to become closed, have been constructed in great numbers. Let us consider, for instance, a ring divided diametrically (fig. 42, *A C*), [1] each half of which attracts the other (§ 103)—that is, tends to diminish the magnetic reluctance of the whole. The variation of the reluctance for a small virtual alteration of the width of the whole gap forms, by well-known mechanical analogies, a measure of the attraction between the two halves of the ring.[2]

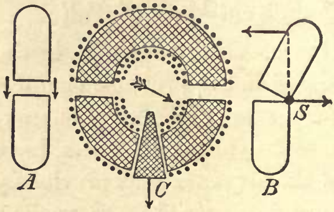

FIG. 42

This attraction, therefore, with a vanishing or a very narrow gap is considerable, but rapidly diminishes as the distance between the halves of the ring increases. The range within which the attraction exhibits or exceeds a prescribed value is, therefore, a small one. Various plans have been devised to get over these defects of the rapid decrease in the attraction, and the small range of the motion, some of which we will mention.

section are taken, with the kind assent of the author, from the latter book. This affords the latest and most exhaustive account of the appliances met with in this special branch of electrotechnics.

[1] In the following figures the ferromagnetic parts in the section are represented by cross shading.

[2] [See J. Hopkinson, the *Electrician*, vol. **33**, p. 100, 1894.]

For instance, the approach can be prevented from taking place freely, as would be represented by the plain arrows of fig. 42, but by suitable guides can be made to take a slanting direction—in the direction of the feathered arrows, for instance. Or the free motion may be increased or be equalised by any of the well-known kinematic arrangements for transmission—different kinds of levers, toothed wheels, &c.; the variations in attraction may also be partially compensated by suitable springs. Another method consists in closing the circuit (fig. 42, *C*), more or less, by means of an iron wedge, which is moved in a direction at right angles to the centroid. By suitably choosing the section of the wedge, an approximately uniform increase of the reluctance may be obtained when the wedge is drawn out in the direction of the arrow. Where no great attraction is required, a better equalisation may be produced by never completely opening the circuit, so that the continuity of the ferromagnetic substance is never entirely broken. The plan of such an arrangement is seen in fig. 42, *B*, where the halves of the ring turn about a joint *S*, and therefore always touch in this point. The magnetic attraction will exert a pretty uniform torque on the upper half of the ring, in its rotation, as represented by the arrows.

Figs. 43, 44, and 45 represent various types of electromagnets used for the most varied purposes, the action of which

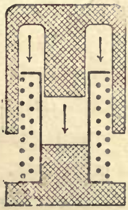

FIG. 43

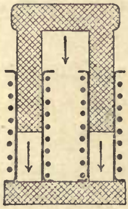

FIG. 44

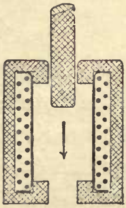

FIG. 45

is at once seen. The range here is tolerably extensive, and can partially be regulated; the attraction is, however, not uniform. These forms are the transition to appliances in which a soft iron core is drawn into a coil, which may be either iron clad or uncovered, to the discussion of which we will now turn.

§ 160. **Small Iron Sphere in a Magnetic Field.**—Let us first investigate generally the mechanical forces exerted by an electromagnetic field on small ferromagnetic bodies in it. For simplicity's sake we will take the case of a small iron sphere, as it offers no preferential direction. In an external field of arbitrary distribution, its magnetisation will, by symmetry, be in the direction of the intensity of the field $\mathfrak{H}_e$. From § 33, the curve of magnetisation of an iron sphere scarcely differs from a straight line through the origin of co-ordinates, the equation of which, since the demagnetising factor of a solid sphere is $4\pi/3$ (§ 30), is the following:

$$(21) \quad . \quad . \quad . \quad \mathfrak{J} = \frac{3}{4\pi} \mathfrak{H}_e$$

and this equation will hold with sufficient approximation up to values of something like

$$\mathfrak{J} = 1500 \text{ C.G.S.}, \qquad \text{that is,} \qquad \mathfrak{H}_e = 6000 \text{ C.G.S.}$$

If V is the volume of the sphere, its magnetic moment may be written

$$(22) \quad . \quad . \quad \mathfrak{M} = \mathfrak{J} V = \frac{3}{4\pi} V \mathfrak{H}_e$$

Now it may be shown that the mechanical force exerted by the field on a very small sphere in a particular direction—for instance, that of the X-axis—has the following component $\mathfrak{F}_x$:

$$(23) \quad \mathfrak{F}_x = \mathfrak{M} \frac{\partial \mathfrak{H}_e}{\partial x} = \frac{3}{4\pi} V \mathfrak{H}_e \frac{\partial \mathfrak{H}_e}{\partial x} = \frac{3}{8\pi} V \frac{\partial (\mathfrak{H}_e^2)}{\partial x}$$

If we now imagine a surface through all points of the field in which the intensity, quite apart from its direction, has a prescribed numerical value, this will form a magnetic *isodynamic surface* on which $\mathfrak{H}_e$, and therefore also $\mathfrak{H}_e^2$, is constant. If we now consider, in the usual way, a group of such surfaces in space, which correspond to an arithmetical series of values of a constant surface parameter (§ 38), the resultant force on the sphere at each point is directed along the perpendicular $\mathfrak{N}$ to the corresponding surface passing through the point, and amounts to

$$(24) \quad . \quad . \quad \mathfrak{F} = \mathfrak{F}_\nu = \frac{3}{8\pi} V \frac{\partial \mathfrak{H}_e^2}{\partial \mathfrak{N}}$$

and the force for a ferromagnetic sphere is in the direction of increasing values of $\mathfrak{H}_e$. We may sum up those considerations in the following:—

II. *In any given field a small ferromagnetic sphere tends always to pass from places of weaker to places of stronger intensity; and this quite independently of the direction of this vector.*

The mechanical force $\mathfrak{F}$ exerted on the sphere further has evidently the scalar potential (§ 39)

$$(25) \qquad \Phi = -\frac{3}{8\pi} V \mathfrak{H}_e^2$$

The above principle was propounded by Faraday on the basis of his experimental investigations. Its mathematical enunciation is due to Lord Kelvin.[1]

§ 161. **Attractive Action of Circular Conductors on Sphere.** Let us apply this fundamental principle to the simple case of a

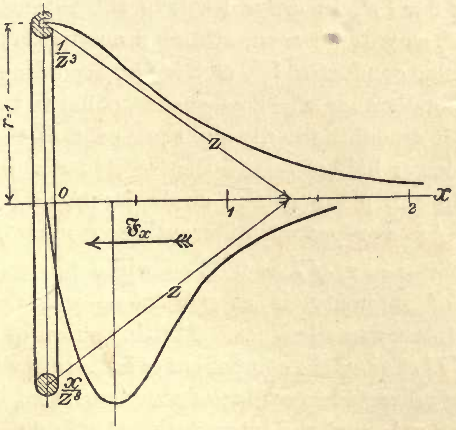

FIG. 46

plane circular conductor carrying the current *1*. Let r be the radius of the circle, x the distance along the axis (fig. 46), $z = \sqrt{x^2 + r^2}$, the distance of a point on the axis from the

[1] Faraday, *Exp. Res.*, vol. 3, series 21, especially § 2418; Sir W. Thomson, Reprint *Electr. and Magnet.* §§ 643–646. This potential Φ of the mechanical force $\mathfrak{F}$ must be carefully distinguished from the magnetic potential Υ (§§ 45, 48); neither are the magnetic isodynamic surfaces directly connected with the ordinary equipotential surfaces.

circumference. The numerical value of the field intensity in a point of the axis (x) [§ 6 C, equation (4)] is

$$(26) \qquad \mathfrak{H}_e = \pm \frac{2\,\pi\, I\, r^2}{z^3} \qquad \text{hence} \qquad \mathfrak{H}_e^2 = + \frac{4\,\pi^2\, I^2\, r^4}{z^6}$$

Let the small iron sphere be restricted to motion along the X-axis, for instance, by being compelled to move without friction along a tube. The component of force then acting upon it amounts, according to equations (23) and (26), to

$$(27) \qquad \mathfrak{F}_x = \frac{3}{8\,\pi} V \frac{\partial\,(\mathfrak{H}_e^2)}{\partial\, x} = \frac{3}{2}\,\pi\, V I^2 r^4 \frac{\partial\, z^{-6}}{\partial\, z}\frac{\partial\, z}{\partial\, x}$$

$$= -\,9\,\pi\, V I^2 r^4 \frac{x}{z^8}$$

In the top half of fig. 46 the function z^{-3} is graphically represented. According to (26) this is to be multiplied by the constant $\pm\, 2\,\pi\, I\, r^2$, in order to have the intensity in a given place. This, as will be seen, attains a maximum in the plane of the circular conductor [$\mathfrak{H}_e = 2\,\pi\, I/r$, according to equation (5), § 6]; at a distance $x = 2\,r$—that is, equal to the diameter of the circle—it amounts to only eight per cent. of that maximum.

In the lower half of fig. 46 the fraction x/z^8 is represented. This is 0 in the plane of the circle itself, and then rapidly increases. As a repeated differentiation shows, it attains a maximum for $x = r/\sqrt{7} = 0{\cdot}38\,r$—while the steepest part of the curve of intensity is at $x = 0{\cdot}5\,r$—and then gradually diminishes to very small values. Multiplication of that function by $-\,9\,\pi\, V I^2\, r^4$ gives the component of force in absolute measure. This is always directed towards the conductor, that is in the sense of increasing values of the intensity of the field; and this independent of its direction. The left half of fig. 46 is omitted, as it is symmetrical with the right half represented.

§ 162. **Attractive Action of Coils on Spheres.**—The field in the axis of a long, uniformly-wound coil (§ 6 D) may be regarded as the superposition of the fields due to individual turns. The transition from the portion of the field in the middle of the coil, which is known to be uniform, towards the outside is represented by a curve, which is like that of fig. 46. The field at the opening attains half the value of its value in the middle,

and then rapidly diminishes. A small iron sphere, with its motion again restricted to the axis of the coil,[1] will, as with the circular conductor, be drawn into the coil. The attractive action is at its maximum near the opening, and then decreases as we approach the region of appreciably uniform intensity, within which a mechanical action is of course no longer exerted on an iron sphere.

An accurate representation of the forces by equations would be as difficult as it would be without object, as they always depend on the particular dimensions of the coil. If we dispense with uniform winding, we may influence as we like the distribution of the field, and therewith that of the attractive force along the axis. This can be attained by altering the pitch of the winding, so that the number of turns per unit length is a variable quantity. If, for example, we desire for any purpose to produce a steady attractive action within a prescribed range on the axis of the coil, then we must have

$$\frac{\partial (\mathfrak{H}_e^2)}{\partial x} = C$$

hence

$$\mathfrak{H}_e^2 = \int C\, dx = Cx + B$$

and

$$\text{(28)} \qquad \mathfrak{H}_e = \pm \sqrt{Cx + B}$$

in which C and B are constants. No general directions can be given as to how the coil must be wound so as to produce a given field. In any given case this must be determined by actual trial, in which the principles laid down may serve as some guide.

For a given position of the sphere the attractive force, according to equation (27), is, *ceteris paribus*, proportional to the square of the current in the coil. Saturation can never be obtained with a sphere under the influence of the field of a coil.

The discussions in this paragraph apply not only to small spheres, but, approximately, also to other pieces of iron, the dimensions of which are nearly equal in all directions, and are

[1] Such a restriction is here necessary as well as with the circular conductor, because the iron sphere, in accordance with Faraday's principle, would otherwise move towards places of higher intensity on the surface of the conductor; that is, over the inner surface of the coil.

small in comparison with those of the coil. But when one dimension preponderates, and is comparable with the length of the coil, the theory is rather complicated, and we are at present compelled to return to empirical investigation.

§ 163. **Attractive Action of Coils on Iron Cores.**—We possess numerous measurements on the attraction of 'short' or 'long' cores—that is, of such as are shorter or longer than the coil. We shall pass over the older researches of Hankel, Dub, von Feilitzsch, von Waltenhofen, &c.,[1] and confine ourselves to the discussion of the more recent systematic ones of Bruger.[2]

His experimental arrangement will be sufficiently clear from fig. 47; the weight of the core in each case was compensated by one sliding weight, and the attraction was measured by another weight for various relative positions of the core and of the coil. Equilibrium is obtained, that is to say, the attraction ceases, when the middle of the core M, supposed to be symmetrical, coincides with the middle of the coil m. That follows from symmetry, as well as from the principle of least magnetic reluctance (§ 158). We determine, therefore, the relative position by the height Y of the middle of the core over the middle of the coil. If the short or long core is moved downwards, the force of suction only attains an appreciable value just before the lower end U of the core enters the upper opening o of the coil. It then rises to a maximum, and falls

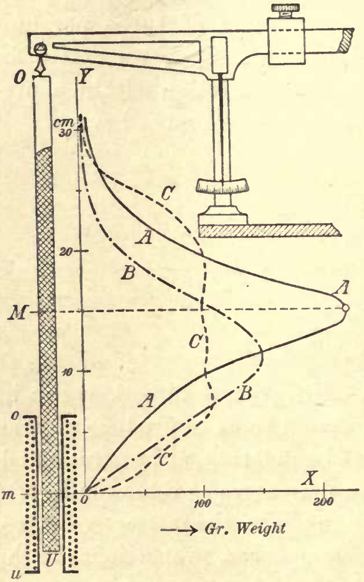

FIG. 47.—$\frac{1}{6}$ scale

[1] G. Wiedemann, *Lehre von der Elektricität*, vol. 3, §§ 651–665, 1883.

[2] Bruger, *Action of Solenoids on Iron Cores of various Shapes.* Inaugural Dissertation. Erlangen, 1886. [See also Mr. Mahon, the *Electrician*, vol. 35, pp. 293, 604; 1895. It would appear possible at present to give a theory describing these actions with more or less approximation.—H. du B.]

off again until the above position of equilibrium is attained. Bruger restricted himself to investigating long coils, and found, in agreement with older statements, that the maximum suction corresponds to about the position in which the lower end of the core U is just on the point of emerging from the lower opening u of the coil (as represented in fig. 47); and this holds for cores the length of which is twice that of the coil.

Bruger used, among others, a coil with the following constants: length $L_s = 13$ cm.; number of turns $n = 266$; field $\mathfrak{H}_m$ in the middle of the coil, per ampere,

$$\frac{\mathfrak{H}_m}{I'} = \frac{0{\cdot}4\,\pi n}{L_s} = \text{about } 30 \text{ C.G.S.}$$

He obtained, for example, with three cores of about 39 cm. ($= 3\,L_s$) length the curves given in fig. 47. Y defines the position of the core, X gives the corresponding suction in gramme-weight. In the diagrams of the curve:

$\overline{AA}$, full curve: cylindrical core; $\mathfrak{H}_m$ = about 180 C.G.S.
$\overline{BB}$, dot and dash curve: double cone; $\mathfrak{H}_m$ = about 180 C.G.S.
$\overline{CC}$, dotted curve: moniliform core; $\mathfrak{H}_m$ = about 250 C.G.S.

For the details we must refer to the paper. Conical cores are used in some arc lamps. The moniliform core gave, as intended, a constant pull over a great range. As long as saturation is excluded, the attraction is roughly approximately proportional to about the square of the current in the coil. Bruger gives curves like those above for various currents. The phenomena for iron-clad coils, as represented in fig. 45, p. 247, are similar; the field then between the inner edges of the shell is more intense and more uniform than in a coil without shell, but outside the coil it diminishes so much the more rapidly, by which the character of the traction curve is somewhat but not materially altered. The case of coils without shells but with a lining of iron is quite different. These do not attract iron cores, but, on the contrary, repel them. Having regard to the principle discussed in § 158, this appears at once intelligible.

§ 164. **Polarised Mechanisms.**—The phenomena are markedly simpler if the core of the coil is not previously magnetised by the current, but has from the outset permanent magnetisation

(§ 46). This case we will discuss. We have seen in § 21 that a mechanical force is exerted on the single end of a magnet in a field, which is in the direction of the latter, and is equal to the product of the intensity of the field into the strength of the end. This may be experimentally realised by bringing one end of a long permanent magnet into the field of the coil, so that the action of the latter on the other end of the coil may be neglected.

In fig. 47, p. 252, let $\overline{UO}$ be a powerful steel magnet, the lower end U of which would move in the direction of the field of the coil or against it, according as its sign is positive or negative. In the middle part of the coil, where the field is uniform, the mechanical force exerted on the end will also be invariable, while it decreases through the openings outwards in proportion to the intensity of the field. It is assumed that the field of the coil is so weak that it exerts no appreciable inductive action on the permanent magnetisation of the steel core.

The arrangement described belongs to the group of what are called *polarised mechanisms*, in which permanent magnets may in any way be used. Two properties present themselves, which in certain circumstances it is desirable to develop as far as possible.

In the first place, a 'bilateral' action, in consequence of which, with currents of opposite directions, actions about proportional to these currents and opposite to one another may be obtained, which is excluded in purely electromagnetic mechanisms.

Secondly, the possibility of attaining greater sensitiveness of the attraction with weak currents—or rather with small changes of current $d\,I$—as results from the following consideration. According to Maxwell's law [§ 103, equation (12)] the attraction is

$$\mathfrak{F} = \frac{S}{8\,\pi}\,\mathfrak{B}^2$$

From this it follows, by differentiation, that the sensitiveness $d\,\mathfrak{F}/d\,I$

$$\frac{d\,\mathfrak{F}}{d\,I} = \frac{S}{4\,\pi}\,\mathfrak{B}\,\frac{d\,\mathfrak{B}}{d\,I}$$

And since $d\mathfrak{H}/dI$ is constant,

$$(29) \qquad \frac{d\mathfrak{F}}{dI} = \text{const.}\ \mathfrak{B}\frac{d\mathfrak{B}}{d\mathfrak{H}}$$

Like the differential $d\mathfrak{B}/d\mathfrak{H}$ itself, the product $\mathfrak{B}\ (d\mathfrak{B}/d\mathfrak{H})$, and therefore the sensitiveness, attains (§ 154) a maximum for a definite value $\mathfrak{B}_1$, which can from the outset be produced by permanent magnetisation, $\mathfrak{J}_1 = \mathfrak{B}_1/4\pi$. The magnetic circuit is best made up of steel and soft iron; the former then furnishes the permanent induction, while the current is allowed preferentially to act on the latter, since the value of $d\mathfrak{B}/d\mathfrak{H}$ for iron is far greater.

In the third place, soft iron, with its high permeability and relatively small hysteresis, may be combined with steel—which, conversely, has low permeability—in such a manner that one of the two properties appears considerably less prominent. This method is analogous to that by which, in optics, pairs of prisms can be combined so as to have either no deviation or no dispersion; and just as achromatic systems can be constructed, it is clear that in the present case it is possible to construct arrangements so that they are almost devoid of hysteresis.

Polarised mechanisms have for long been used, for example, in duplex telegraphy, in Hughes's printing telegraph, in many telephones, and other apparatus. Magnetic circuits consisting of steel and iron parts of various section (used as shunt) have been proposed by Abdank-Abakanowicz, Evershed and Vignoles, and John Perry to compensate or diminish hysteresis in the above manner, which, as is well known, forms a chief source of error in measuring instruments.[1]

§ 165. **Electromagnets with large Lifting-power.**—After Sturgeon, in 1825,[2] had constructed the first electromagnet, attempts were made to obtain great statical lifting power or rather load ratio. At the present day these attempts are of but subordinate importance, though machines have of late been constructed—for instance, for welding iron plates, as well as for appliances for transmitting power, or for brakes—in which

[1] Abdank-Abakanowicz, *Electrician*, vol. **32**, p. 93, 1893. Vignoles, *ibid.* p. 166. Field and Walker, *ibid.* p. 186.

[2] Sturgeon, *Trans. Soc. of Arts*, vol. **43**, p. 38, 1825.

electromagnetism is used exclusively for maintaining a statical attractive force. In Chapter VI. (§§ 109, 110) we have already drawn conclusions from Maxwell's law of magnetic stress which are to be considered as the bases for the construction of the kind of electromagnet here contemplated. As the attraction is, *cet. par.*, proportional to the square of the induction, it is most important to increase that quantity to the value of 20,000 C.G.S. units, which is conveniently attained in practice, and which represents a pull of about 16 kg.-weight per square centimetre (§ 103). In order to attain this result with as small a number of turns as possible, the reluctance of the magnetic circuit, which, with such electromagnets, must always be closed, must be as small as possible—that is, its shape must be tolerably compact.

As, moreover, with a given induction the attraction is proportional to the cross-section, the latter must be as great as possible. At the same time, it may in some circumstances be advisable to somewhat diminish the section in the neighbourhood of the bounding surfaces of the gap, from the principle laid down in § 109, according to which the attracting force for a given induction, apart from leakage, is inversely proportional to the section. This diminution of section, as was remarked (*loc. cit.*), is not, however, to be driven too far; in the first place, because the validity of that principle is ultimately restricted by leakage, and, in the second place, because any such 'throttling' of the magnetic circuit increases the reluctance. The ends, consisting of two or more separate surfaces, of each of the two parts of the magnetic circuit, must be carefully prepared, and ground or polished, so that the reluctance of the resultant joint is as small as possible (§ 151). The softest wrought iron, of high permeability, must be used. To divide it is for the present purpose not only without advantage, but is even injurious, owing to the increased magnetic reluctance. The question as to the possible load-ratio has been discussed in § 110, so that we need not here recur to it.

§ 166. **Description of some Types of Electromagnets**.—The oldest form resembles the thin permanent horseshoe magnets then most in use (fig. 48). The coiling usually consisted of two long coils on each limb. From the considerations in the previous paragraph, a compacter form, such as fig. 49, deserves

the preference.[1] The investigations of Joule on the lifting power of magnetic circuits (§§ 105, 111) led him to the construction of very powerful electromagnets, which are even now met with, and are known as 'Joule electromagnets' (fig. 50). They are distinguished by their length, and may be constructed from

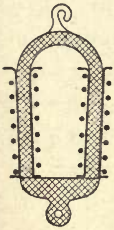

FIG. 48

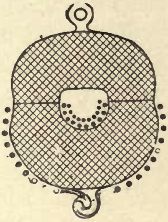

FIG. 49

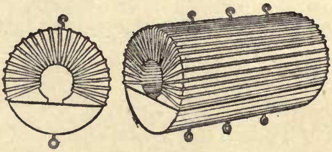

FIG. 50

a stout iron tube cut lengthwise—for instance, a gun barrel. Instead of the horseshoe shape, electromagnets are now much used in which the coils are on straight limbs connected through a rectangular iron yoke, the keeper being usually of the same shape.

With these are connected what are called 'clubfooted' electromagnets, in which one limb is coiled, while the other is useful only in closing the magnetic circuit; the latter, according to the plan of Nickles, may also be effected by two or more magnetomotively 'dead' iron bars, which are symmetrically arranged about the coil, and the total section of which is at least equal to that of the core inside. These various bars may, lastly, also be merged into a cylindrical iron shell; this leads us to the type of the iron-clad 'Bell Magnets' introduced by Guillemin and Romershausen. The iron shell is connected with the core by means of an iron disc, the keeper being of the same form. Similar electromagnets, with several iron shells and coils fitted into each other, have been constructed by Camacho, but hardly appear to offer theoretical advantages.

The ironclad forms represent a transition to the very effective electromagnets in which the conductor is embedded in

[1] The keeper is not usually covered with wire; but from the considerations in § 94 and § 95, it scarcely matters where the requisite number of ampere-turns is placed, that is where the magnetomotive force chiefly originates.

narrow grooves in the ferromagnetic substance itself. To these belong the older electromagnets of Roberts and Joule, the construction of which is sufficiently obvious from figs. 51 and 52. An arrangement described by Forbes and Timmis may finally

FIG. 51

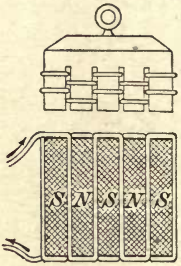

FIG. 52

be mentioned, which is represented in fig. 53, and may be regarded as a modification of Radford's, in which the windings are spiral instead of circular, as in the former. The electromagnets last mentioned have been chiefly constructed with a view of transmitting motion by means of the friction set up, and also for brakes on carriage wheels as represented in fig. 53. It may be mentioned in this connection that what is called 'magnetic friction' is essentially due to the greater pressure between the surfaces in contact. The question whether and in how far the coefficient of friction itself is affected by magnetisation, possibly in consequence of molecular processes on the surface, must for the present be considered as an open one.[1]

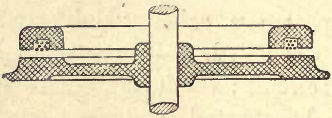

FIG. 53

C. *Electromagnets for producing Strong Fields*

§ 167. **Review of the usual Types.**—The problem of producing strong magnetic fields is of great interest for experimental physics; the investigation of many phenomena, especially those in which the square of the intensity of the field comes into play (§§ 203, 229), can only be successfully carried further by using

[1] For further details we may refer to the work of Silv. Thompson. See also G. Wiedemann, *loc. cit.* §§ 358–371.

the strongest possible fields. The question as to the rational design of electromagnets for this purpose has, nevertheless, scarcely been considered. We shall discuss this question, therefore, somewhat more in detail than the electromagnetic arrangements hitherto mentioned, and the more so as we shall derive from it an interesting application, as well as a confirmation of the theory of Chapter V. (§§ 171–173).

Of the types of construction which have been empirically arrived at, perhaps those in most extensive use are copied from horseshoe magnets; like these (§ 166), they consist of two vertical coiled limbs, the lower ends of which rest on a massive iron yoke, while to the upper ends pole-pieces of suitable form

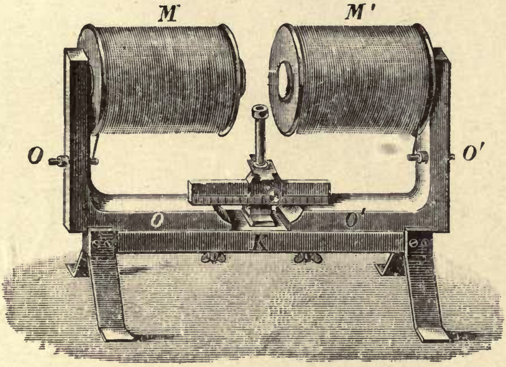

FIG. 54

may be fitted. With vertical electromagnets the legs are usually very long, in order to bring in the requisite number of turns; owing to their being so near one another there is considerable leakage between them, which diminishes the available flux of induction between the pole-pieces.

The electromagnets of Ruhmkorff's construction are widely known, and are as effective as they are convenient to use (fig. 54). The two angular iron pieces $\overline{O\ O}$ and $\overline{O'\ O'}$ may be moved horizontally, and may be clamped in any given position, by which the space between the pole-pieces may be conveniently adjusted; to these may be screwed the horizontal cores, which for magneto-optical experiments are bored. The position of the coils M and M' is in any case far more rational than in the vertical

electromagnets (§ 173). The objection may be raised that the circuit $\overline{OKO'}$ is too weak magnetically as well as mechanically; the consequence of this is that, on the one hand, its magnetic reluctance is greater than necessary, and, on the other, the two angle pieces bend under the influence of magnetic tractive force, by which the distance of the poles is diminished. The magnetomotive force might moreover be considerably increased by coiling the lower connecting piece K.[1]

In this respect, the magnetic circuit of the electromagnet represented in fig. 55, and used by Ewing and Low in their

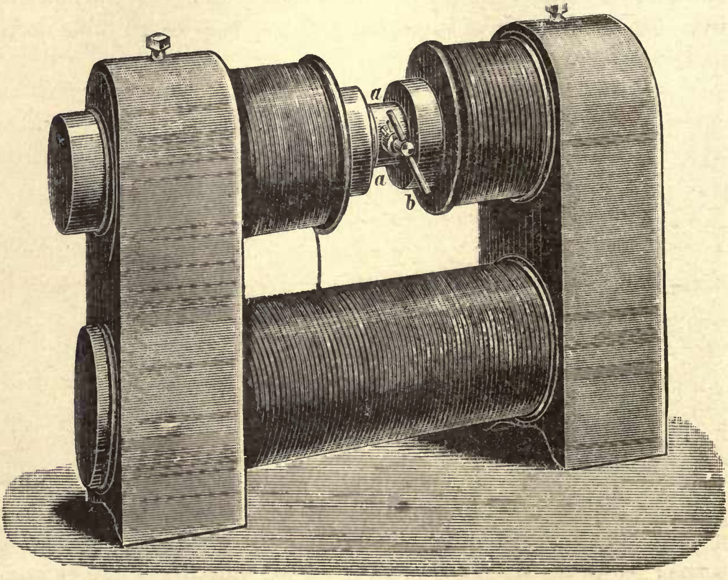

Fig. 55

isthmus method (§ 217), is better arranged. It is true that this is at the cost of convenience of manipulation, which in Ruhmkorff's construction is unexcelled; this drawback might, however, be remedied by suitable mechanical arrangements. The electromagnet represented could be excited with 64,000 ampere-turns;

[1] [These objections might be met by suitably designing connecting pieces of greater cross-section and rigidity; the adoption of modern 'cast steel' of high permeability would probably render such a type the best and cheapest for producing all but the very strongest fields.—H. du Bois.]

in this way Ewing and Low obtained an intensity $\mathfrak{H} = 24{,}500$, and an induction $\mathfrak{B} = 45{,}350$ C.G.S., the highest which had at that time been obtained in soft iron. As regards the field intensities observed in the air, it follows from the published statements that until recently values above 28,000 to 30,000 C.G.S. units had not been reached.

§ 168. **Principles of Design.**—There seemed no *a priori* reason why the production of still stronger fields should be impossible. The author accordingly attempted the construction of an electromagnet for this purpose, and in this attempt he was guided by the following considerations. The first point is to begin with the production of as high a value as possible of the flux of induction, which then, by 'throttling' the magnetic circuit by means of suitable pole-pieces, may, as it were, be concentrated as described below (§ 175). Accordingly the magnetic reluctance which, especially owing to the unavoidable air-space between such pole-pieces, cannot be indefinitely diminished, must be overcome by as great a number of ampere-turns as possible (§ 173).

In all the electromagnetic apparatus and machines we have hitherto discussed, and indeed in the great majority of such, it was sufficient from the nature of the case to consider only the first two stages of the process of magnetisation. But in the present problem the third, or stage of saturation, alone need be considered. In consequence of this, and of the circumstance that considerations of economy, certainty of working, facility of repair, and the like, are of less account in the present case, the conditions of construction are, to some extent, different. The discussion of § 95 showed that the field of the coil finally tends to completely direct and dominate the distribution of the vectors in the magnetic circuit; hence the coiling must be such that there is everywhere, and especially between the pole-pieces, a field in the desired direction—that is, tangential to the centroid of the magnetic circuit. In such an arrangement leakage will ultimately decrease as the saturation increases, and the induction-tubes so gained will be utilised; this result was confirmed by experiment (§ 173). As regards the shape of the ferromagnetic substance, the theoretical conditions already mentioned are best satisfied by a toroid divided radially. In other respects, the points discussed in § 141 for the construction of the frames of

electromagnets may be referred to; they only hold, however, in the present case *mutatis mutandis*. Starting from the principles developed, the author has constructed an electromagnet which we shall proceed to describe.[1] The experiments made with this, to be discussed afterwards, must be considered as a confirmation of the validity of the principles followed in its design.

§ 169. **Description of the Electromagnet.**—As it was found that, with proper tooling appliances, the construction of a heavy iron toroid was neither more difficult nor more costly than building up a frame of several parts, the former was decided on. Fig. 56 represents the electromagnet in $\frac{1}{15}$ the natural size, partly in outline, and partly in section. In the notation of fig. 15, p. 109, $r_1 = 25$ cm., $r_2 = 5$ cm.; hence $L = 157$ cm., and $S = 78{\cdot}5$ sq. cm. At S the toroid is divided tangentially to the inner circle. A horizontal sliding motion is here introduced, so that the right side of the toroid can be displaced in reference to the left by means of a handle or wheel G, and thus the upper air-space Z be conveniently adjusted. The slide is so constructed that the break of continuity of the ferromagnetic substance is as small as possible. However, with the preponderating magnetic resistance of the air-gap Z a few joints are not of much account (§ 152).

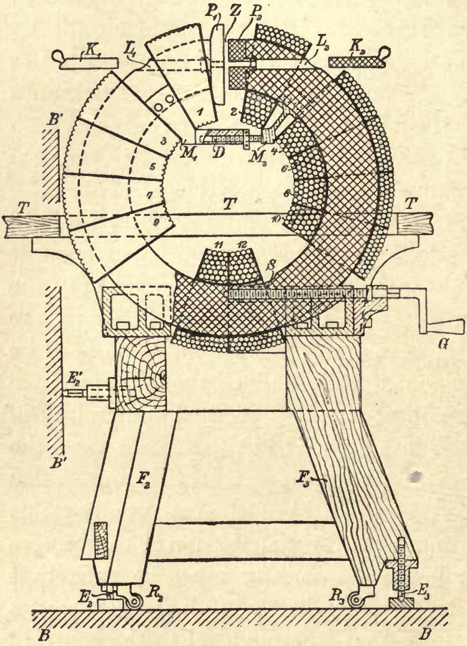

FIG. 56.—$\frac{1}{15}$ scale

In order to prevent any bending in consequence of the con-

[1] § 169, du Bois, *Wied Ann.*, vol. **51**, p. 507, 1894; *Elektrotechn. Zeitschrift*, vol. **15**, p. 203, 1894.

siderable tractive force, a brass holder $\overline{M_1 D M_2}$ is fitted, which, by means of a screw, can be adjusted to the width of the gap at any time. By using flat pole-pieces, separated by a narrow slit, the tractive force is so great that only discs of metal placed between can resist it.[1] The perforation L_1, L_2 in the direction of the field allows of magneto-optical observations if desired, but iron plugs K_1 and K_2 are usually inserted, since an unnecessary increase of reluctance in this place is not desirable.[2] The toroid rests on bronze bearings, which, in turn, are supported by a massive wooden tripod F_1, F_2, F_3, provided with rollers R_1, R_2, R_3 and levelling screws E_1, E_2, and E_3. The table TT serves for placing on it accessory apparatus. The axis of the field may, by tilting the whole apparatus, be set vertical, which is desirable for certain experiments.

§ 170. **Coils of the Electromagnet.**—We have hitherto omitted to discuss general rules for winding and plans for connecting because in every special case this is simply determined by pre-existing conditions in a comparatively simple manner.[3] It may perhaps be mentioned that the traditional rules for the use of batteries—that is, sources of current, whose electromotive force and internal resistance are assumed constant (which, however, seldom occurs)—have less interest at the present time. In most cases we are now concerned with self-regulating dynamo machines, street mains, or accumulators—that is, sources of current which furnish a more or less constant difference of potential, and in using which a definite limit of current may generally not be exceeded.

[1] Assuming a tension of 16 kg.-weight per sq. cm. (§ 103), the total pull is $\mathfrak{F} = \mathfrak{Z} S = 16 \times 78{\cdot}5 = 1250$ kg.-weight, whereas the entire weight of the whole electromagnet is 270 kg. Suitable discs are provided, 1, 1, 1, 2, 5, 10, 10 mm. thick like a set of weights.

[2] When the air-space Z is not too small, it has little effect, as observation shows, whether the poles are filled with iron cores or not, as the reluctance of the air then preponderates (§ 175). Similar statements are made by Leduc (*Journ. de Physique* [2], vol. **6**, p. 239, 1887). As to the properties of hollow iron cores in general, reference must be made among others to von Feilitzsch, *Pogg. Ann.*, vol. **80**, p. 321, 1850; Silv. Thompson, *loc. cit.*, pp. 86, 184; Leduc, *La Lumière électrique*, vol. **28**, p. 520, 1888; Grotrian, *Wied. Ann.*, vol. **50**, p. 705, 1893; du Bois, *ibid.*, vol. **51**, p. 529, 1894 (see also note 1, p. 235).

[3] Compare Silv. Thompson, *loc. cit.*, chapter vi., where this question is thoroughly discussed for steady currents; in chapter vii. follows a discussion

In the present case the former amounted to 108 volts, and the latter to about 50 amperes. Each individual coil comprises a sector of the circumference of 20°; its 200 turns have about 0·2 ohm resistance when warm. If the 12 coils are arranged in series, they have, accordingly, 2·4 ohms resistance, and they cover 240°—that is, two-thirds of the circumference. With that total resistance the difference of potential, 108 volts, produces a current of 45 amperes; this corresponds to a magnetomotive force of 108,000 ampere-turns, or 136,000 C.G.S. units. Dividing the latter number by the perimeter $L = 157$ cm., we get 860 C.G.S. for the mean intensity of the field of the coils. Of these only about 380 C.G.S. is to be considered as a direct inductive agent. With the iron actually used[1] the magnetisation attained is 1600 C.G.S. The excess of intensity (480 C.G.S.) serves exclusively for counteracting the demagnetising action. If the value of magnetisation given is to be maintained, a demagnetising factor up to

$$N = \frac{480}{1600} = 0{\cdot}3$$

is admissible. This, in fact, is its value with the widest air-spaces which occur in use—that is, with pointed pole-pieces.

The power necessary for exciting the maximum effect of the electro-magnet is

$$108 \times 45 \text{ volt-amperes} = 4860 \text{ kilo-watts} = 6{\cdot}5 \text{ H P}$$

Its greatest self-inductance, with closed magnetic circuit, if we disregard the demagnetising action of the sliding guides, as well as of other joints, which, however, may scarcely be neglected, may by § 153 (eq. 8) be calculated as follows:

$$\Lambda = \frac{2\,n^2 S}{r_1}\,\frac{d\mathfrak{B}}{d\mathfrak{H}_e} = \frac{2\,.\,(2400)^2\,.\,78{\cdot}5}{25} \cdot 5000 = 180 \text{ henries}^2$$

of the winding and connecting of coils for variable currents as required in electromagnetic mechanism, when as rapid an action as possible is an essential requirement.

[1] This was the same brand as that from which the toroid described in § 83 was turned, and the normal curve of which is represented in fig. 21, p. 131.

[2] From the curve of ascending reversals (o), fig. 21, p. 131, we find the maximum value of the differential quotient $d\mathfrak{J} / d\mathfrak{H}_e = 400$ (for $\mathfrak{H} = 1$ C.G.S.); from this follows $d\mathfrak{B} / d\mathfrak{H}_e = 4\pi\, d\mathfrak{J} / d\mathfrak{H}_e = 5000$ [§ 154, equation (9)].

The corresponding maximum value of the time-ratio is then $\theta = A/R = 180/2{\cdot}4 = 75''$. For instance, the time was observed with the magnetic circuit closed for various values of the steady current I'_e—that is, of the field of the coil $\overline{\mathfrak{H}}_e$—which elapsed after making the (variable) current I' until it attained 90 per cent. of its steady value; firstly when the apparatus was previously demagnetised (T_1), and secondly when there had been a previous magnetisation (T_2) in the opposite direction:

I'_e=0·1 amp.;	$\overline{\mathfrak{H}}_e$= 2 C.G.S.	I'=0·09 amp.;	T_1=98″;	T_2=185″
0·2	4	0·18	74″	128
1·0	20	0·90	17″	—
2·0	40	1·80	10″	—
5·0	100	4·50	5″	—

These numbers speak for themselves (compare fig. 40, p. 242).[1]

With high self-induction a sudden break or even reversal of the current is out of the question, for the extremely high electromotive force which would thereby be produced would endanger the insulation, or at least produce too strong a spark on breaking. In the present case the simplest remedy was adopted —that is to say, a 'carbon switch.'[2] Ballistic experiments were obviously out of the question, owing to the great self-induction, and recourse was therefore necessary to another method of investigation.

§ 171. **Method of Investigation.**—The electromagnet has a number of flat poles like P_2 in fig. 56, p. 262; if these are screwed in on each side, the apparatus represents a divided toroid with adjustable air-gap. The author has made measurements, in order to test experimentally by another method the conclusions of Chapter V.[3]

In the first place, the mean intensity of the field of the coil

[1] With a non-inductive resistance of about 40 ohms in circuit, the above time amounted, on the contrary, *ceteris paribus*, to only fractions of a second; this depends on the fact that the impressed electromotive force in the sense of § 153 is then not constant, but at the beginning assumes a far higher value than corresponds to the steady current.

[2] There are a number of other methods of avoiding injurious sparking, or perforation of the insulation. Silv. Thompson, *loc. cit.*, devotes a special chapter (xiv.) to this subject.

[3] There exist in addition the following more or less extended series of measurements on electromagnets of Ruhmkorff's form (fig. 54, p. 259), which, however, have been made according to other principles and methods: Stenger, *Wied. Ann.*

was calculated from the current I', measured in amperes; by using all the 12 coils ($n = 2400$) [§ 72, equation (1)], it was

$$(30) \quad . \quad . \quad \overline{\mathfrak{H}}_e = \frac{2\,n\,I'}{10\,r_1} = 19{\cdot}2\,I'$$

The total difference of magnetic potential $\Delta\,\Upsilon_t$ between the flat poles provided with narrow bore-holes was determined by a magneto-optical method to be described in the next chapter (§ 199). If d is again the width of the slit, the potential difference $\Delta\,\Upsilon_i$ is obtained by subtracting from the above that portion $\mathfrak{H}_e d$ which arises from the direct action of the coil; hence the value deduced from the observations is

$$(31) \quad . \quad . \quad \Delta\,\Upsilon_i = \Delta\,\Upsilon_t - \mathfrak{H}_e\,d$$

On the other hand the theory of Chapter V. [§ 80, equation (19)] gives

$$(32) \quad . \quad . \quad \Delta\,\Upsilon_i = \overset{E}{\underset{A}{\Upsilon_i}} = \frac{4\,\pi\,\overline{\mathfrak{J}}\,d}{\nu}$$

Experiments were now made; I, for $d = 1{\cdot}13$ cm.; II, for $d = 2$ cm.; corresponding to $d\,/\,r_2 = 0{\cdot}226$ and $0{\cdot}400$ respectively; that is to say, two values which are also found in Table V, § 89. Hence the corresponding curves (4) and (5), fig. 22, p. 133, could be used, which, according to Lehmann, represent ν as a function of $\overline{\mathfrak{J}}$; only so far, it is true, as his observations extend, and the reciprocity of ν and $\mathfrak{n}$ holds with sufficient approximation—that is, up to values of about $\overline{\mathfrak{J}} = 1400$ C.G.S. Taking as basis the formula for the demagnetising factor [§ 80, equation (111)]

$$(33) \quad . \quad . \quad \overline{N} = \frac{2\,d}{\nu\left(r_1 - \dfrac{d}{2\,\pi}\right)}$$

the magnetisation curves for the two gaps could be obtained, and from them, according to equation (32), the dotted curves which in fig. 57 theoretically represent $\Delta\,\Upsilon_i$ as a function of $\mathfrak{H}_e$. The observed values of $\Delta\,\Upsilon_i$ are moreover plotted, and the individual points connected by straight lines; for further details the paper quoted must be referred to.

vol. **35**, p. 333, 1888; Leduc, *Journal de Physique* [2], vol. **6**, p. 238, 1887, and *La Lumière électrique*, vol. **28**, p. 512, 1888; Czermak and Hausmaninger, *Wiener Berichte*, vol. **98**, p. 1142, 1889.

§ 172. **Confirmation of the Theory.**—The agreement between observation and theoretical calculation is satisfactory, as seen in fig. 57, and affords the theory developed in Chapter V. an additional support, quite independent of the experimental confirmation there given. Curves I and II at first coincide; this is due to the fact that the reluctance of the air is then so great compared with the remaining reluctance, that virtually the entire magnetomotive force is engaged in overcoming the former, and hence, for a given value of it, the increase of potential in the gap is sensibly equal to it, and this independently of the width of the slit, provided this is not too small. The behaviour of the electromagnet when flat poles are used is defined by that quantity $\Delta\Upsilon_i$; the mean self-induced intensity in the air-gap is found by dividing $\Delta\Upsilon_i$ by the width of the gap. Hence, from what has been said, up to semi-saturation that vector is sensibly inversely

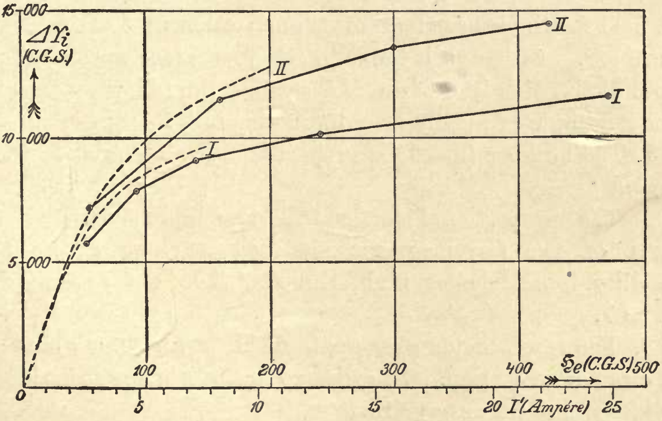

FIG. 57

proportional to the width of the gap. It does not follow that the field is infinitely great for infinitely narrow gaps, for then the above relation would not hold; the upper limit of the field is rather the maximum value of $4\pi\mathfrak{J}$ attainable in practice—that is, about 20,000 C.G.S. (§ 103). If to this be added the relatively small intensity of the field of the coil, the total available intensity is obtained

$$\overline{\mathfrak{H}_t} = \overline{\mathfrak{H}_i} + \overline{\mathfrak{H}_e}$$

It may be remarked that the results given are not to be

regarded as a simple confirmation of the experiments described in §§ 83–90, merely because here the toroid is relatively thicker ($r_2 / r_1 = 0{\cdot}200$) than in the previous case ($r_2 / r_1 = 0{\cdot}112$). The fact that the coefficients of leakage of fig. 22, applied to the present special case, give correct values, forms among others a farther support for the correctness of the view, firstly, that the gap as such determines the numerator, and secondly that the rest of the toroid rules independently the denominator of the demagnetising factor.

§ 173. **Investigation of Leakage.**—In consequence of this, H. Lehmann's empirical formula for leakage [§ 90, equation (30)] acquires more general interest; it was

$$(34) \quad . \quad . \quad . \quad . \quad \nu - 1 = 7\frac{d}{r_2} \qquad \left[0 < \frac{d}{r_2} < \frac{1}{2}\right]$$

and held for the lower stage of magnetisation up to about semi-saturation. Although it holds in the first place specially for a toroid of Swedish iron of circular section, the following assumptions hold approximately even more generally.

1. The number $(\nu - 1)$ is proportional to the 'relative width of gap.'

2. The proportional factor will depend but little on the nature of the ferromagnetic material, as long as its permeability is sufficiently high, and can therefore generally be put = 7.

3. The result holds also approximately for gaps which are not circular in section. In order to transform the equation for the latter case we observe that

$$S = \pi r_2^2; \text{ hence } \frac{1}{r_2} = \sqrt{\frac{\pi}{S}}$$

Introducing this value into the above equation we have

$$(35) \qquad \nu - 1 = 7\sqrt{\pi}\frac{d}{\sqrt{S}} = 12{\cdot}5\,\frac{d}{\sqrt{S}} \qquad \left[0 < \frac{d}{\sqrt{S}} < \frac{1}{4}\right]$$

In this shape the equation may in many cases prove useful.

In order to gain an insight into the leakage when flat pole-pieces are used, a compass was arranged in the 'second

position' (§ 191) at a distance of two metres from the electromagnet; its deflections gave an approximate measure of the absolute amount of leakage. It was found that the latter, as was to be expected, increased with the width of the gap. As the magnetising current was increased, the leakage increased rapidly from zero to a maximum (for gap I about 0·1 C.G.S.), which was attained for a current of about 3 amperes; the leakage then gradually decreased, until at 45 amperes it was only a fifth to a third part—according to the width of the gap—of the above value. This ultimate decrease is evidently still more pronounced with the 'relative leakage'—that is, indirectly with the coefficient of leakage—and this forms a simple support for Lehmann's result (§ 88, II), which was stated in § 123 to be a 'crucial' experiment, in opposition to the views there put forth.[1]

The two 'polar coils,' 1 and 2 (fig. 56, p. 262), exert a predominant influence in this direction. When they were switched out of circuit, the difference of magnetic potential, especially with the strongest currents, sank by as much as 20 per cent., while the leakage was increased 4 to 6 times as much. Both these coils, then, in a certain sense, play the part assigned to them (§ 168) of keeping together the flow of induction, and of utilising the stray induction tubes, which would otherwise be lost.[2] The rest of the coils, as special experiments showed, have the object of furnishing ampere-turns—that is, magnetomotive force—and in this respect all of them are sensibly equivalent.

The greater part of the stray induction-tubes will be confined to the immediate neighbourhood of the gap. By using 'plate poles' (like P_1 in fig. 56, p. 262) many more of them will therefore be utilised.

§ 174. **Theory of Conical Pole-pieces.**—In order to obtain the most intense fields, pointed pole-pieces, sometimes of very peculiar shape, have long been used, since with flat poles, as observed above, the limit of 20,000 C.G.S. can hardly be exceeded. The most suitable form of such pointed poles was first theoreti-

[1] The disturbing action at a distance will thus be comparatively small, precisely with the strongest currents, and in any case will be considerably less than that of great electromagnets of less simple form. This fact represents an advantage in large laboratories, which must not be under-estimated.

[2] This may be the chief reason for the superiority of Ruhmkorff's electromagnets over those with vertical limbs (§ 167).

cally investigated (1888) by Stefan, and almost simultaneously by Ewing and Low.[1]

They make a similar assumption to that in § 76—namely, that the distribution of magnetisation is uniform throughout the cross-section up to within the pole-pieces; that is, its direction is parallel to the X-axis (fig. 58). This assumption corresponds here also to the ideal limiting case of absolute saturation which in reality is not at all reached (§ 89). The expression obtained represents, therefore, limiting values, which we will again denote by the index ∞. Now consider pole-pieces of the shape represented in fig. 58. On the above assumption the end-elements on the conical surfaces will have by § 49 the surface density

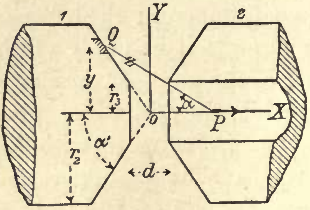

FIG. 58

$$(36) \qquad \mathfrak{J}_\nu = \mathfrak{J} \cos (\mathfrak{J}_\nu, X) = \mathfrak{J} \sin a'$$

Now, by the theory of gravitation, it can easily be shown that in the point 0 the self-excited intensity, in so far as it depends on the conical surfaces on each side, is given by the following expression:

$$(37a) \qquad \overset{(a)}{\mathfrak{H}}_{i\,\infty} = 4\,\pi\,\mathfrak{J}_\infty \sin^2 a' \cos a' \log_e \frac{r_2}{r_3}$$

For other points of, along as well as somewhat beside, the X-axis, the intensity may be expressed by spherical harmonics. The differential quotient of the trigonometrical expression in (37a), that is,

$$\frac{d\,(\sin^2 a' \cos a')}{d\,a'} = \sin a' \cos^2 a' \;(2 - \tan^2 a')$$

is obviously zero for

$$a' = \tan^{-1} \sqrt{2} = 54^\circ\,44'$$

which expresses that, theoretically, the maximum intensity is obtained when the semi-angle of the cone is 55° in round numbers. This result is quite independent of the distribution of the end-elements on the surface of the cone. Moreover, as Ewing and

[1] Stefan, *Wiener Berichte*, vol. **97**, p. 176, 1888; *Wied. Ann.* vol. **38**, p. 440, 1889. Ewing and Low, *Proc. Roy. Soc.* vol. **45**, p. 40, 1888; *Phil. Trans.* vol. **180**, pp. 227–232, 1889. See also Czermak and Hausmaninger, *Wiener Berichte*, vol. **98**, p. 1147, 1889.

Low observe (p. 231, *loc. cit.*), the reluctance of the air-gap will be greater the smaller the angle of the cone. The magnetisation, and indirectly the intensity of the field, are thereby diminished, so that those experimenters consider that in round numbers 50° is the best semi-angle.

In case the pole-pieces have either an axial bore, of radius r_3 (fig. 58, right side), or are connected by a cylindrical neck (what is called an 'isthmus,' § 217) of such radius, end-elements do not occur except on the surfaces of the cone. But in the case of truncated cones (fig. 58, left), the action of a pair of base-surfaces, of radius r_3, at the distance $2\,r_3 \cot a'$ (fig. 16, p. 112), must be added. We obtain their mean intensity if we divide the expression for the magnetic difference of potential [§ 76, equation (16)] by that distance. This portion then becomes

$$(37b) \quad \overset{-(b)}{\mathfrak{H}}_{i\,\infty} = 4\,\pi\,\mathfrak{J}_\infty \left(1 + \frac{1}{2}\tan a' - \sqrt{1 + \frac{1}{4}\tan^2 a'}\right)$$

The sum of (a) and (b) gives the limiting value of the total self-excited intensity of the field. Ewing and Low showed, further, that if uniformity of the field in axial and in transverse direction is specially desired at the expense of high intensity, it is better to choose a smaller semi-angle than the above—namely, $a' = 39° \, 14'$. For further details as to these instructive theoretical investigations the papers cited must be referred to. Czermak and Hausmaninger treated also the case in which the two summits of the cones do not coincide in the point 0, as has been assumed above.

§ 175. **Experiments with Truncated Cones.**—With the electromagnet described above the author has investigated how far the above assumptions and results hold good in practice. For this purpose a pair of truncated cones were turned step by step smaller, so that the angle a' became gradually less. The intensity of field was each time determined by the U-tube method (§ 204), and its highest value was, in fact, found for an angle $a' = 60°$. As it is only a question of a maximum, it does not really matter practically as long as it is between $63° > a' > 57°$.

No appreciable advantage was found in having the pole-pieces concave. The observed value of the field was always several

thousand C.G.S. less than the theoretical. The following measurements were obtained:

for $r_3 = 2{\cdot}5$ mm. 36,800 C.G.S.
for $r_3 = 1{\cdot}5$ mm. 38,000 C.G.S.[1]

It follows, from the formulæ of the previous paragraph, and is confirmed by experiment, that high intensity of the field is only attained at the cost of its extent. For many experiments, however, an extent of several sq. mm. is sufficient, or else the methods of investigation must be adapted to satisfy this condition.

The bore-holes, which are indispensable in magneto-optical experiments, produce relatively more weakening of the field the wider they are as compared with the distance of the faces. This weakening is, however, less than would follow from the equations in § 174, since there is a kind of internal leakage from the edge of the openings towards the axis. The external leakage and action at a distance when truncated cones are used are similar to that described for plane poles.

What has been stated in the previous section may be summed up by saying that a ring electromagnet of manageable size, with truncated conical pole-pieces of 120° aperture, enables us to have fields up to, say, 40,000 C.G.S. over an extent of some square millimetres. To exceed this to any material extent could at present be only accomplished by an undue expenditure of means out of all proportion with the end in view. That follows already from the formula given in which log (r) comes in; while the weight and cost of an electromagnet are determined rather by the third power of its linear dimesions.

D. *Inductors and Transformers*

§ 176. **Discussion of Mutually Inducing Coils.**—We have already explained the principal manifestations of self-induction by the example of a uniformly coiled toroid, either closed or divided

[1] This field would produce in a piece of thin soft iron between the truncated cones the approximate induction equal to

$$\mathfrak{B} = 38{,}000 + 4\pi \times 1750 \doteq 60{,}000 \text{ C.G.S.}$$

to which would correspond a total longitudinal stress equal to (§ 103)

$$\mathfrak{Z} = \left(\frac{60{,}000}{5000}\right)^2 = 144 \text{ kg.-weight per square cm.}$$

radially (mean radius r_1, perimeter $2\pi r_1 = L$, section S). Besides that primary coil 1 (resistance R_1, number of turns n_1, self-inductance Λ_1), let there be a secondary one 2 (R_2, n_2, Λ_2), also wound uniformly on the toroid, as would be the case with the experimental ring described in § 83. Taking the case of such a pair of mutually inducing coils, we will explain as briefly as possible the most important facts in mutual induction, in so far as they are essential for understanding what follows.

To a variation of the flux of induction $\mathfrak{G}_1$ produced by the primary current corresponds an electromotive force E_{i2} in the secondary, which, from equation (6), § 153, will have the following value:

$$(38) \qquad E_{i2} = \frac{d\,(n_2\,\mathfrak{G}_1)}{d\,T} = \frac{\partial\,(n_2\,\mathfrak{G}_1)}{\partial\,I_1}\,\frac{d\,I_1}{d\,T}$$

The partial differential quotient of $(n_2\,\mathfrak{G}_1)$ in respect of the primary current I_1, which here occurs, is called, in analogy with the definition of § 153, the mutual inductance Ξ_{12} between the coil 1 and the coil 2. We find, in the present case,

$$(39) \qquad \Xi_{12} = \frac{4\,\pi\,n_1\,n_2\,S}{L}\,\frac{d\,\mathfrak{B}}{d\,\mathfrak{H}_e} = \frac{n_2}{n_1}\,\Lambda_1$$

If now the parts played by each coil in the phenomenon are interchanged, so that 2 now acts inductively on 1, we have, in an analogous manner,

$$(40) \qquad E_{i1} = \frac{(d\,n_1\,\mathfrak{G}_2)}{d\,T} = \frac{\partial\,(n_1\,\mathfrak{G}_2)}{\partial\,I_2}\,\frac{d\,I_2}{d\,T} = \Xi_{21}\,\frac{d\,I_2}{d\,T}$$

in which, again, the mutual inductance is

$$(41) \qquad \Xi_{21} = \frac{4\,\pi\,n_2\,n_1\,S}{L}\,\frac{d\,\mathfrak{B}}{d\,\mathfrak{H}_e} = \frac{n_1}{n_2}\,\Lambda_2$$

§ 177. **Mutual Induction.**—Now it is obvious from the above that $\Xi_{12} = \Xi_{21}$. Hence this quantity may be called, without further definition, the 'mutual inductance' of the two coils.[1]

From (39) and (41) we see, directly,

$$\text{that for } n_1 \gtreqless n_2 \text{ we have } \Lambda_1 \gtreqless \Xi \gtreqless \Lambda_2$$

[1] Mutual as well as self inductance has the dimensions of a length, and like this is to be expressed in henries (compare note, p. 237).

Further, for any given value of $\mathfrak{B}$ we have

$$\Xi = \sqrt{\Lambda_1 \Lambda_2} \qquad (42)$$

These conclusions are based on the assumption that in (39) and (41) S as well as $d\,\mathfrak{B} \,/\, d\,\mathfrak{H}_e$ are identical; that is, in other words, the common flux of induction threading through the two coils is the same. When a ferromagnetic core is used, this is always the case with sufficient approximation, since the induction tubes outside it produce no appreciable difference (compare, however, § 183).[1]

As the mutual inductance Ξ differs from the differential quotient $d\,\mathfrak{B} \,/\, d\,\mathfrak{H}_e$ by a constant factor only, it will be sufficient to refer to § 154 and fig. 36, *B*, p. 225, where the character of that differential quotient is completely discussed. For closed magnetic circuits there can be no question of a constancy for Ξ, any more than there can for Λ; for unclosed magnetic circuits, on the contrary, that assumption is here also the more admissible, the greater the value of the demagnetising factor.

By means of an arrangement quite similar to that here assumed, in which, however, each of the two coils was severally placed on one half of a ring, Faraday, as is well known, discovered, in 1831, the existence of induction phenomena.[2] Almost simultaneously, and quite independently of him, J. Henry[3] also carried out his fundamental researches in this department, more especially on 'extra current,' as it was called; he already introduced the expression 'self-induction.' The kind of apparatus used by these experimenters had undergone a long course of development,[4] until, after the lapse of half a century, a division into two classes was made, *inductoriums*, or *induction coils*, and *transformers*, of which the first has mainly a scientific interest; in the decade which has elapsed since that separation, the latter class has from its technical importance undergone an

[1] The circumstances are quite different with two coils without cores, in any given relative position; such cases, however, may be disregarded for our present purposes.

[2] Faraday, *Exp. Res.* vol. **1**, Series **I.** § 27 (Plate I, fig. 1).

[3] J. Henry, *Collect. Scient. Writings*, vol. **1**, p. 73 *et seq.*; Silliman, *Americ. Journ.* vol. **22**, p. 403, 1832.

[4] G. Wiedemann, *loc. cit.* vol. **4**, §§ 409–430; Fleming, *loc. cit.* vol. **2**, chapter i.

extraordinary development, both in the theoretical relations and in the practical details of construction.

§ 178. **Action of Induction Coils.**—In an induction apparatus, in the narrower sense of the word, extremely high electromotive forces are induced in a secondary coil, which has a great many turns, by making or breaking the primary current. These are mostly used to produce electric discharges of various kinds between the ends of this coil. The discharge on making—corresponding to the gradual rise of a given primary current, as described in § 155, starts a quantity of electricity in a closed secondary coil, equal to the break discharge which takes place in the opposite direction; because in both cases the same total variation $\delta\mathfrak{G}$ of the flux of induction and the same resistance are concerned [§ 64, equation (14)]. Although, then, the time-integral of the secondary electromotive force is in both cases the same, its maximum value is far greater on breaking the primary current; for this process, though never instantaneous, is, however, considerably more rapid than the gradual increase of the current up to a large fraction of its steady value, which takes place on making (§ 15). In the case therefore of spark discharges, only the 'break-spark' passes across wide spaces of air; with the same distance of the electrodes the 'make-spark' cannot traverse the air-path. It follows from this, that it is of paramount importance to break the primary as rapidly as possible; for this purpose the following means are more especially applied.

In the first place, contact-breakers are used, which reduce as much as possible the primary spark due to the self-induction of the primary coil, which prevents an absolutely instantaneous break of the current. Experience shows that in this respect mercury contact-breakers are best, working under insulating liquids, such as alcohol or petroleum. (Note 2, p. 265.)

In the second place, it is usual to adopt Fizeau's plan of interposing a condenser near the break. This diminishes the injurious spark by partially storing up the primary discharge. For any given primary coil the most suitable capacity, that which most effectually extinguishes the primary spark, while it prolongs the secondary one, is to be found by experiment. This probably depends on one of the phenomena of resonance which

by the brilliant discoveries of Hertz have recently claimed prominent attention. The oscillating primary discharge is then, as it were, thrown back by the condenser, and produces a primary current in the opposite direction; in this way it is obvious that longer secondary break-sparks are formed than when the primary current merely vanishes without altering its direction.

§ 179. **Magnetic Circuit of Induction Coils.**—It is not sufficient that the break or the reversal be as rapid as possible; it is also necessary that the corresponding variation of the flux of induction in the core shall directly follow it.

Let us assume at first that the primary current only falls to zero; for the induction in the secondary coil the corresponding decrease of the flux of induction—that is to say, the evanescent magnetisation—denoted in § 149 by $\mathfrak{J}_E$—is of paramount importance. Let us imagine, for example, a closed core of good soft wrought iron; with an intensity of $\mathfrak{H}_e = 20$ C.G.S., let the magnetisation $\mathfrak{J}_M = 1000$ be attained; as the retentivity in this case might easily amount to 90 per cent., $\mathfrak{J}_R$ would be $= 900$, hence $\mathfrak{J}_E$ would only equal 100 (see § 149 and fig. 36, *A*, p. 225). The case is different with an unclosed core, the demagnetising factor of which we suppose to be, say, $\overline{N} = 0{\cdot}02$, corresponding to ($\mathfrak{m} = 45$, Table I, p. 41); taking the coercive intensity $\mathfrak{H}_C = 2$ C.G.S., we have $\mathfrak{J}_R = \mathfrak{H}_C / \overline{N} = 100$, and hence, $\mathfrak{J}_E = 900$. At the same time a demagnetising intensity $\mathfrak{H}_i = \overline{N}\,\mathfrak{J}_M = 20$ will have to be compensated, so that the field of the primary current must be doubled, that is $\mathfrak{H}_e$ must be increased to 40 C.G.S. From the purely magnetical point of view, there would scarcely be any object in making the dimension-ratio of the core less than 45, and thereby needlessly increasing the demagnetising action.

But if the primary current does change its direction owing to the action of the condenser, a closed core appears again more advantageous, provided that the opposing primary field attains at least the coercive intensity; an inspection of fig. 36, *A*, p. 225, shows the correctness of this contention. On the other hand, the smaller the demagnetising factor the greater is the value of the differential quotient $d\,\mathfrak{B} \,/\, d\,\mathfrak{H}_e$ (§ 154); the self-induction of the primary coil is hereby increased in a manner

difficult to calculate, but which is scarcely desirable. We cannot at present be said at all to possess a comprehensive rational theory of the induction coil, probably because these apparatus have not hitherto had any extensive practical applications. One great difficulty is that we have no sufficient knowledge of what takes place when the current is broken and produces a spark with or without condenser; such a theory must necessarily be mainly based on that knowledge.

The most practical, efficient and best known forms of induction coils are produced by a few workshops; their empirical principles of design are doubtless based on experience, and are partly kept secret.[1] We must, therefore, be content with referring to the fact that the cores generally consist of bundles of wire, the dimension-ratio of which varies between 15 and 20 (that is, $0{\cdot}12 > \overline{N} > 0{\cdot}08$, Table I, p. 41); the thickness of the wire is usually about 1 mm. With reference to the most recent investigations on eddy currents as occurring in transformers (see § 187), it might be desirable to have thinner iron wires for the cores, with their much more rapid magnetic variations, and without any corresponding disadvantages inherent to their use.

§ 180. **Simultaneous Differential Equations of Transformers.** The first transformers of induction coils, introduced about a decade ago, had a general resemblance to the induction coils then in use, and accordingly had an unclosed core of thin iron wire. Following the plan of Zipernowsky (1885), forms with closed magnetic circuit were soon reverted to; in these Faraday's ring, or the uniformly double-wound toroid discussed in § 176, became the prototype for by far the greater number of transformers now made. Accordingly the magnetic characteristic (§ 96) of the transformer circuit becomes very simple, since neither an air-gap nor leakage has to be considered (§ 183). On the other hand, the expression for the magnetomotive force is somewhat more complicated on account of the action of two coils; for obviously the following equation must be always satisfied:

$$(43) \qquad M = L(\mathfrak{H}_1 + \mathfrak{H}_2) = 4\pi(n_1 I_1 + n_2 I_2)$$

[1] See du Moncel, *Notice on Ruhmkorff's Apparatus*, 5th edition, Paris, 1867. Most induction coils are based on this apparatus with but slight modifications.

Moreover, the Hopkinson function (§ 97) now becomes

$$(44) \quad . \qquad M = F_H(\mathfrak{G}) = Lf\left(\frac{\mathfrak{G}_1 + \mathfrak{G}_2}{S}\right)$$

where L is the perimeter of the centroid, and the indices 1 and 2 refer to the primary and secondary coil respectively. Hence, the equation of the magnetic circuit becomes

$$(45) \quad . \qquad 4\pi(n_1 I_1 + n_2 I_2) = Lf\left(\frac{\mathfrak{G}_1 + \mathfrak{G}_2}{S}\right)$$

Using the notation of § 176, and in connection with the discussion there given, the differential equation of the primary coil then becomes

$$(46) \qquad \frac{d\,n_1(\mathfrak{G}_1 + \mathfrak{G}_2)}{d\,T} + R_1 I_1 = \Lambda_1 \frac{d\,I_1}{d\,T} + \Xi \frac{d\,I_2}{d\,T} + R_1 I_1 = E_{e1}$$

it asserts that the sum of the three electromotive forces corresponding to self-induction, mutual induction and ohmic resistance are at each moment equal to the impressed electromotive force E_{ei}; the latter is a periodic function of the time, and in a certain number of cases, it may with sufficient approximation be regarded as a sine function.[1] The circuit external to the secondary (resistance R_2') being supposed devoid of self-induction and capacity, and therefore offering no source of E.M.F., we have

$$(47) \qquad \frac{d\,n_2(\mathfrak{G}_1 + \mathfrak{G}_2)}{d\,T} + (R_2 + R_2')\,I_2 = \Lambda_2 \frac{d\,I_2}{d\,T} + \Xi \frac{d\,I_1}{d\,T} + (R_2 + R_2')\,I_2 = 0$$

We cannot expect to arrive at a general solution of these two simultaneous differential equations with the two variables I_1 and I_2, since the inductances Λ_1 Λ_2 and Ξ in a closed magnetic circuit are, as our former reasonings show, neither constant nor unambiguous (§§ 154, 177). The only plan is to consider, in the first place, an 'ideal transformer' in which

[1] The time-curve of an electromotive force furnished by an alternator depends on the form of its pole-pieces and coils; moreover, the self-induction of the armature, &c. shows a certain tendency to partially annul the deviations from the sine form. A discussion of the curve of periodic electromotive forces given by some of the usual alternators is given by Fleming, *loc. cit.* vol. 2, pp. 446–475.

these functions are supposed to be constant; by assuming that the primary electromotive force is sinusoidal, it is then found that the primary as well as the secondary current, and therefore also the secondary electromotive force E_2, are sine functions showing differences of phase among each other which can be calculated; it is, in fact, obvious that the above differential equations can be satisfied by such functions. From the behaviour of the ideal transformer, conclusions, more or less pertinent, as to that of a transformer with an iron core may be drawn. This method depends on Maxwell's general method of mathematically treating any given inducing pairs of coils.[1]

J. Hopkinson[2] has pursued a somewhat different plan, which depends on the identity, apart from leakage (§ 183), of the flux of induction embraced by the two coils (see note 2, p. 240); in consequence of this, the differentials may be eliminated in the first mode of writing equations (46) and (47), by multiplying the former by n_2 and the latter by n_1, and subtracting; we then obtain the following equation:

$$(48) \qquad . \qquad n_2 E_{e1} = n_2 R_1 I_1 - n_1 (R_2 + R_2') I_2$$

If this is combined with equation (45) of the magnetic circuit, and if some terms are disregarded, we also get data for approximately judging the action of a transformer.

§ 181. **Action of an Ideal Transformer.**—We shall first discuss the case in which the resistance R'_2 of the external secondary circuit is infinite—that is, it is broken and the transformer therefore runs 'empty.' There being no current in the secondary coil, it cannot react on the first, and the latter behaves like a simple induction coil, the properties of which are discussed in § 153 *et seqq.* It follows from the equations there given that, with a sufficiently high self-inductance—that is, time-ratio—the primary current is considerably choked, and moreover a retardation of phase χ as against the primary electromotive force will be shown, which approaches the value $\chi = (1/2\pi) \tan^{-1} = 1/4$; in the last case we have to deal with what is called a 'wattless' primary current, which represents

[1] Maxwell, *Phil. Trans.* vol. **155**, I, p. 459, 1865. See also Mascart and Joubert, *Electricity and Magnetism*, vol. **1**, p. 593; vol. **2**, p. 834.

[2] J. Hopkinson, *Proc. Roy. Soc.* vol. **42**, p. 85, 1887; *Reprint of Papers*, p. 182. New York, 1893.

no electrical energy.[1] With an ideal transformer running empty, the flux of induction has the same phase as the primary current which produces it alone, while the secondary electromotive force induced by the variations of the former is in all circumstances retarded by a quarter of a phase in reference to it.

If now the secondary current is made, and the transformer is thus loaded, the phenomena are more complicated; we shall confine ourselves to mentioning the chief points. The self-induction of the primary coil is apparently diminished by that action, so that the primary current is so much the stronger and its phase gains the more on that of the primary electromotive force, the greater the load. The electrical power supplied to the primary coil, which must exceed the power corresponding to the sum of the load, and of the loss of efficiency in the transformer, at the same time increases sufficiently to meet the demand. In case the self-induction of the secondary coil is small, and its external circuit free from induction, the phase of the current in it is but little behind the electromotive force; in respect of the primary current it usually shows a retardation of phase of almost $\frac{1}{2}$—that is, the two currents with a full load have usually almost opposite phases, and therefore magnetise the core in opposite directions. The ratio of the mean primary, to the mean secondary electromotive force, is called the *transformation-ratio* or the coefficient of transformation $\mathfrak{p}$; for the ideal transformer hitherto considered this is

$$(49) \quad . \quad . \quad \frac{E_1}{E_2} = \mathfrak{p} = \frac{n_1}{n_2} = \sqrt{\frac{A_1}{A_2}}$$

and this relation holds very nearly for practical transformers. As, therefore, the electromotive forces are proportional to the corresponding number of turns, it is clear that the products $n_1 I_1$ and $n_2 I_2$—that is, the primary or secondary ampere-turns respectively—will be but little different; in consequence of this, their algebraical sum, occurring in equation (45), or, as they have

[1] The mean electrical power of a sinusoidal alternating current is known to be equal to half the product of the maximum values of the electromotive force, and the current, multiplied by cos $2\pi\chi$, and would therefore vanish for $\chi = 1/4$. For running empty such a condition is of course to be aimed at; that it is, however, only approximately realised is a weak point in the use of transformers permanently inserted in at any rate the primary circuits, the secondary coils of which run empty for days together, or are but little loaded.

opposite signs, their numerical difference will only be small in comparison with each of them singly. The curve which represents the secondary electromotive force (or the potential at the terminals[1]) as a function of the secondary useful current, may be called the 'total (or external) characteristic' of the transformer, just as it was with the dynamo (§§ 126, 127). This, however, is not the place to discuss it, as it has by no means so simple a connection with the magnetic characteristic (equation 44) as in the case of dynamos. The processes described are made plainer by plotting a *transformer diagram*, which represents the four quantities E_1, I_1, E_2, I_2 as periodical functions of the time; many suitable methods have in recent times been devised for this purpose (see fig. 59, p. 284). After these brief remarks we proceed to discuss the modifications of the ideal processes caused by allowing for magnetic saturation and hysteresis as well as for leakage.

§ 182. **Influence of Saturation and of Hysteresis.**—As regards saturation, the inductances Λ_1, Ξ, and Λ_2 diminish considerably even before it is attained. Phenomena occur in consequence of this which are similar to those produced in a simple induction coil (§ 157) by saturation. The behaviour of the transformer, briefly speaking, would approximate to that of a coreless one; as this, however, must absolutely be avoided, the induction $\mathfrak{B}$ in good transformers at the present time is scarcely allowed to exceed the value 6000 to 7000 C.G.S., at which the inductances have exceeded their maximum, and just begin to decrease (fig. 41, p. 243). This value, which is usual, in practice amounts to less than half that often reached in dynamos or electromotors (§ 126). Accordingly, since even an incipient saturation is nowadays never allowed to occur, there is scarcely any interest in further discussing its influence, which, however, with the older transformers is in many ways characteristically observable.

In the second place, hysteresis produces considerable differences of the coefficients of induction for ascending or for descending magnetisation in the magnetic circuit of transformers. It will be seen from the transformer diagram obtained by

[1] In the ordinary winding of transformers, the potential at the terminals moreover usually differs by less than 1 per cent. from the electromotive force.

experiment (fig. 59, p. 284) that the curve of the primary current is chiefly affected by it, which, as we have already seen in the case of the simple induction coil, loses the symmetry of the ascending and descending branches, and in some circumstances acquires a perfectly irregular form, which, however, recurs periodically. As scarcely more than two to three times the coercive intensity are required to produce the induction $\mathfrak{B} = 7000$ in soft iron, the retardation of the zero-point of the ($\mathfrak{I}, T$)- or of the ($\mathfrak{B}, T$)-curve, in respect of the ($\mathfrak{H}, T$)-curve, will be considerable (compare § 157 and fig. 36, C, p. 225); while, according to the foregoing paragraphs, the resultant field $\mathfrak{H}$ is already retarded in respect of I_1 by the action of the secondary current. A very considerable displacement of the ($\mathfrak{B}, T$)-curve in respect of the (I_1, T)-curve results from all this. The zero-points of the former, as appears from many transformer diagrams, are about in the middle, between those of the curves of the primary and the secondary current.

According to Ferraris, the action of hysteresis may be compared with that of a 'dead' secondary coil supposed to be added.[1] Such a one would also retard the ($\mathfrak{B}, T$)-curve, and, more particularly, like hysteresis, would cause a deleterious dissipation of energy into heat. The latter may, however, be diminished as much as possible by keeping the induction within the narrow limits mentioned. This is an additional reason in itself for working with a low degree of saturation. For a given range of induction the loss of power by hysteresis is, as stated in § 148, proportional to the volume of the iron and to the frequency, which is to be remembered in predetermining these quantities.

§ 183. **Influence of Leakage.**—Leakage does not occur at all in the typically shaped and coiled transformer which has formed the basis of the above considerations. The fields due to the currents in each coil, owing to their almost exactly opposite phase (§ 181), act in opposition to each other; but this always produces a resultant periodic field, which, if both coils are uniformly wound, will also be peripherically uniform.

[1] Ferraris, *Mem. R. Acc. di Scienze*, Torino [2] vol. **37**, p. 15, 1885, and vol. **38**, 1887. The action of eddy currents may obviously be considered from the same point of view; the aggregation of their paths may evidently be regarded as a separate secondary coil.

The resultant induction has, therefore, also uniform peripheral distribution. The flux of induction will thus, without any leakage, be confined to the core, and be surrounded by both coils in the same manner. In that case only, from § 177, will the equation

$$\Xi = \sqrt{A_1 A_2}$$

be satisfied. This condition is to be aimed at in all transformers. It is not, however, practicable to wind the primary and secondary coils uniformly together, for in that case the electrostatic capacity becomes excessive, and an adequate insulation is impossible.

If, in the other extreme case, the opposing coils are wound separately on the two halves of the toroid, as in Faraday's ring (§ 177) and in some experiments of Oberbeck (§ 92), there is obviously considerable leakage. For transformers an interdiate course is adopted, by arranging the primary and secondary parts of the coils alternately as regularly as possible. There is, nevertheless, always a certain amount of leakage. This can then be measured by a coefficient of leakage $\nu' > 1$, which is defined by the equation

(50) . . . $$\nu' \Xi = \sqrt{A_1 A_2}$$

This evidently has an entirely different meaning from the coefficient ν previously introduced (§§ 78, 128), and is a periodic function of time. It may be proved that chiefly the symmetry of the (E_2, T)-curve is destroyed by leakage, as appears from transformer diagrams, plotted with an unfavourable arrangement of the two coils—that is, when the leakage is considerable.

§ 184. **Transformer Diagrams.**—In order further to explain the processes described, a typical transformer diagram by Ryan and Merritt[1] is represented in fig. 59, p. 284. Of the great number of diagrams plotted by these experimenters we only reproduce that for 'no load' (A) and for 'full load' (D). In

[1] H. J. Ryan, *Trans. Amer. Inst. Electr. Engin.*, vol. 7, p. 25, 1890; *The Electrician*, vol. **24**, p. 263 and vol. **25**, p. 313, 1890. Fleming gives a large number of such diagrams in his book; *loc. cit.* pp. 446–478. They are also sometimes called, though less briefly, the indicator-diagrams of a transformer.

the transformer examined, $n_1 = 675$, $n_2 = 35$; hence, the theoretical transformation-ratio [equation (49), § 181] $\mathfrak{p} = n_1/n_2$

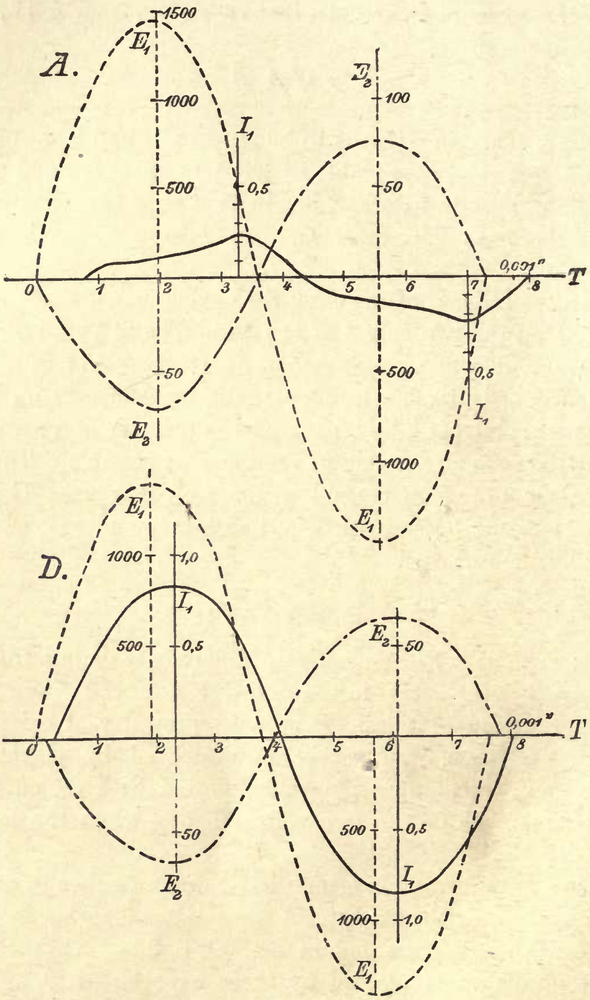

FIG. 59

$= 19{\cdot}3$; the mean length $\overline{L}$ of the closed magnetic circuit was 30·8 cm., the mean section $\overline{S} = 63{\cdot}3$ sq. cm., the volume $V = 2050$ cub. cm. The remainder of the chief data are given

in Table VIII, columns B and C of which represent intermediate loads.[1]

TABLE VIII

Values measured	A No load	B Intermediate loads	C	D Full load	Unit
Period τ	7·3	7·3	7·6	7·6	$\frac{1}{1000}$ sec.
Frequency $N = 1/\tau$ . . .	138	138	132	132	Per second
Primary E.M.F. E_1 . . .	1025	1053	1050	1040	Volts
Primary current I_1 . . .	0·14	0·20	0·39	0·63	Amperes
Power supplied A_1 . . .	96	159	389	608	Watts
Secondary E.M.F. E_2 . .	54·5	52·3	51·0	49·3	Volts
Secondary current I_2 . .	0	1·26	5·83	1065	Amperes
Useful output A_2	0	64	301	525	Watts
Efficiency $g = A_2/A_1$ . . .	0	41	77	87	Per cent.

In these diagrams

1. Dotted curves represent primary electromotive force E_1;
2. Full lines ,, ,, currents I_1;
3. Dot-and-dash ,, secondary electromotive force E_2.

Owing to the small number of secondary turns, and to the fact that in the experiments the external resistance consisted of 1 to 10 incandescent lamps in parallel, the entire circuit may be regarded as a non-inductive one. Hence the curve of its current is identical with the (E_2, T)-curve but for the ohmic resistance as a factor, and was therefore omitted from the diagrams. The summits of the curves are marked by lines of ordinates, which enable a judgment to be formed as to the relations of symmetry. Along those lines the scales to the corresponding curves are placed. After the foregoing discussion the transformer diagrams hardly require further explanation.

§ 185. **Core and Shell Transformers.**—The typical ring transformer which we have hitherto discussed, besides its theoretical simplicity, also offers advantages for practical use, but would be difficult to wind and to repair. Accordingly, a series of modifications of this fundamental type have been developed, which, in conformity with Kapp's nomenclature, are usually called *core transformers*.

[1] The values given for the periodic quantities are, as usual, the square roots of mean squares.

The reverse of this arrangement is a ring formed of the primary and secondary copper conductors, which, in turn, are wound with a sufficient quantity of iron wire. From this type a number of practical designs are derived, which are usually known as *shell transformers.*

These are the two chief classes of transformers, to which most forms may be assigned. In the course of the last few years a great variety of designs have been proposed, and have more or less come into practical use. A sharp distinction can scarcely be made between core and shell transformers, seeing that many forms of them are, as it were, a transition between the two classes.

Given the two coils, which, from what has been stated, are packed together as closely as possible in order to avoid leakage, and which have approximately equal weights of copper, the mutual and self-inductance will be a maximum, when the magnetic circuit is *closed*, of as large a section as possible, and of the softest wrought-iron. In this way the efficiency which can be transformed for the given pair of coils is a maximum; against this advantage, the disadvantage that, on account of the great volume of iron, the loss of power by hysteresis is, *ceteris paribus*, a maximum, does not come into consideration from the purely magnetic point of view.

§ 186. **Magnetic Circuit of Transformers.**—The form of the magnetic circuit is far simpler with transformers than with dynamos, as it is essentially governed by the following rule: 'The immediate vicinity of the coil, in all places where an appreciable field is produced, ought to be filled up with soft iron sufficiently divided.'

[1] By far the greater number of modern transformers have, as stated in § 180, a closed magnetic circuit; there are only a few which have an unclosed one. Besides the points of view discussed in the text, in what is still an open question as to the advantages of both arrangements—other considerations, more of an economical nature, come into play, upon which we cannot here enter. For further details as to transformers, the following works and papers must be referred to: Fleming, *loc. cit.*; Kittler, *loc. cit.* 1st edition, vol. **2**, §§ 177–235; Uppenborn, *Geschichte der Transformatoren*, München, 1889; Blakesley, *Alternating Currents*; Silv. Thompson, *Dynamo-Electric Machinery*, 4th edition, chap. xxv., London, 1892; J. Hopkinson, *Reprint*, pp. 148–216; Ferraris, *La Lumière électrique*, vol. **10**, p. 99, 1885, and vol. **27**, p. 518, 1888.

It will be sufficient to explain by a single example how this rule may be satisfied. It may be premised that the magnetic circuit is filled either with thin iron wire, or, as is most usual, with very thin sheet iron. This is stamped to patterns corresponding to the cross-section of the air-space about the coils which is to be filled. The pair of coils of a number of the best known transformers is, for instance, coiled in an O shape; the lines of intensity therefore are in planes at right angles to the plane of the figure. E-shaped iron laminæ are now packed up parallel to those planes, so that the tongue of the E is stuck through the O, and the flaps are again put together on the other side; such a process is obviously susceptible of a great number of variations. It will be sufficient in this respect to refer to the works cited above.

A magnetic circuit like that described can obviously be considered either as core or as shell; the core may further be regarded as a magnetic circuit divided twice through the shell. Multifold magnetic circuits in the sense of § 145 occur in polyphase (rotary) transformers; for the sake of simplicity we have tacitly restricted ourselves to two-phase currents in the above; the considerations apply also in given cases to three-phase or polyphase currents.

§ 187. **Eddy Currents. Screening Action.**—In our previous discussions we have expressly disregarded parasitic eddy currents, since by sufficient division they may be theoretically brought below any assignable limit, in contradistinction to hysteresis, which is not affected thereby. In good transformers, the dissipation of energy due to the former cause is therefore but small, compared with that due to the latter. The present question has been examined in detail by J. J. Thomson,[1] and by Ewing. According to these researches, the heat developed by eddy currents for a given volume of iron is, *ceteris paribus*, nearly proportional to the square of the thickness of the iron laminæ—as long as this is less than 1 mm.—and to the square of the range of intensity; it also increases with the frequency as well as with the electrical

[1] J. J. Thomson, *The Electrician*, vol. **28**, p. 509, 1892; Ewing, *ibid.*, p. 631; Fleming, *loc. cit.*, vol. **2**, pp. 485–490 and 535–538. [Also J. Hopkinson and E. Wilson, *Phil. Trans.* vol. **186**, A, p. 93, 1895.]

conductivity of the iron, according to a law which is represented by complicated hyperbolic functions.

The magnetic *screening action* also which eddy currents exert on the interior of the bodies in which they occur, is also thoroughly investigated in the place quoted; we shall confine ourselves to giving the practical result: that with 100 periods per second (note, p. 220 and § 148) the iron must not be thicker than $\frac{1}{3}$ mm. With transformers used for that frequency this theoretical result agrees with good practice; for with smaller frequencies this upper limit is raised.

We refer, in addition, to what was said in § 148 about building up armatures. This is the place to point out the analogy between the action of the transformer and of the dynamo. In the secondary coil of the former, and in the armature of the latter, electromotive forces are induced by varying magnetic conditions; in the dynamo the motion relative to the field-magnets produces these variations; in the transformer, on the contrary, the primary current, which itself is variable, produces them.

With a fixed transformer a sufficient loss of heat is not so easily attained as with a rapidly rotating armature, so that in the former case four or five times as great a cooling surface is required as in the latter. A moderate increase of temperature of the core of transformers—say to 90°—is not without advantages. For hysteresis is then less (§ 148), and the loss by eddy currents, owing to diminished electrical conductivity, is so too; while the magnetisation of the iron under these circumstances shows no appreciable decrease up to that temperature.

CHAPTER X

EXPERIMENTAL DETERMINATION OF FIELD-INTENSITY

§ 188. **General Introduction.**—The complete determination of a magnetic field embraces on the one hand a topographical survey of its distribution (Chapter III.) within the region in question, and on the other hand the numerical evaluation of the intensity at a given place, which then also gives that in other parts of the field; the latter problem is by far the more important, as, in general, we are dealing with fields which are either quite uniformly distributed throughout a certain region, or may be approximately considered so.

In the present chapter we shall review the various methods of measurement; in so doing we shall give only the chief features of the older, more or less classical methods, which may be assumed to be known,[1] while several newer methods, less generally understood, will be discussed more in detail. In so far as 'electromagnetic' fields are concerned—that is, such as are exclusively produced by electrical currents—it will of course, if possible, be simpler to calculate the distribution, and the absolute value of the field from the dimensions of the conductors, and the easily measurable current; the corresponding formulæ for the cases of most frequent occurrence are given in §§ 5, 6. In many cases, however, the calculation meets with insuperable analytical difficulties; any such field is, however, subject to the laws of distribution in §§ 44 and 45. The same holds also for fields due to rigid magnets, the distribution of which was discussed in §§ 47–49.

When the calculation is seen to be impracticable, as is

[1] See, among others, F. Kohlrausch, *Leitfaden der prakt. Physik*, 7th edition, Leipzig, 1892; Heydweiller, *Hülfsbuch für elektr. Messung.*, §§ 62–77, Leipzig, 1892; Mascart and Joubert, *Electricity and Magnetism*, vol. 2, §§ 1139–1188, London, 1888.

almost always the case in considering ferromagnetic bodies, recourse must be had to experimental methods; which of these is to be preferred depends on the special circumstances of the case; that is on the accessibility, the extent, and the order of magnitude of the field; and also whether its direction is horizontal or not; whether an absolute or a relative, an approximate, or an exact measurement is intended.

The elements necessary for forming a judgment will appear in what follows.

§ 189. **Distribution of Magnetic Fields.**—The distribution of a magnetic field in two dimensions may, as already stated (§ 4), be represented by means of iron filings. The magnetic diagram, as it is called, obtained in this way not only gives a clear representation of the course of the lines in the plane chosen, but it also gives an approximate idea as to the relative value of the intensity in various places; as the lines are closer, the higher is that value. Lindeck[1] has represented a number of such figures, one of which is reproduced in fig. 60. This is the case of a field in the meridional plane of a circular conductor conveying a current, the position of which appears from the blank places at the top and bottom. The lines of intensity close to the conductors form circles around them, while the field in the middle of the circular conductor is pretty uniform. This figure agrees with the distribution theoretically calculated for this case (§ 6, C).

A method, which is more accurate, though, at the same time, more troublesome, than the use of iron filings, and which gives the lines of intensity in three instead of in two dimensions, consists in following them with a small movable magnetic needle, proceeding from point to point always in its own direction.[2] A small cross-piece $\overline{EW}$ fixed at right angles to the needle gives, according to Searle, the direction of the corresponding equipotential surface. The value of the intensity may often be deduced from the frequency of the vibrations, since the square of the frequency is proportional to that value [see equation (I) in the next article]. When using permanent magnetic needles the superposed induced magnetisation constitutes

[1] Lindeck, *Zeitschrift für Instrumenten Kunde*, vol. 9, p. 352, 1889.

[2] C. Hering, *Electrical Engineer*, vol. 6, p. 292, 1887.

a considerable source of error, especially in intense fields. Hence, in many cases permanent magnetism is dispensed with,

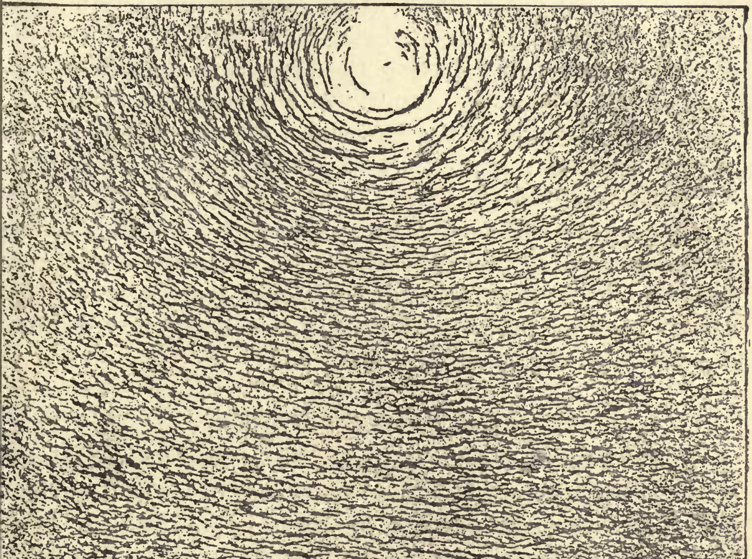

FIG. 60

and a soft iron needle is used. In so far as its magnetisation, in the first stage, is approximately proportional to the intensity of the field, which, from § 33, is nearly the case for not too long

needles, the frequency of the vibrations in this case offers a proportional direct measure of the numerical value of the intensity.

The method of investigating the distribution of a field least open to objection, but which, at the same time, is very tedious, is that by means of an exploring coil (§§ 2, 4). From the position of maximum induction, the direction of the field may be deduced; and from the quantity of electricity set in motion, the numerical value of the intensity. For the details of this method, which is very seldom used, we may refer to the sections on ballistic methods (§§ 195, 196; see also Chapter XI., § 208).

To complete this account we may mention a few instruments in this connection, viz. the *declinometer*, *inclinometer*, and *local variometer*.[1] As these, however, are almost exclusively used for special work on terrestrial magnetic measurements, a detailed description need not be given here.

A. *Magnetometric Methods*

§ 190. **Plan of Gauss's Method.**—The accurate measurement of the absolute value of the horizontal component of a uniform field was first made possible by the classical method which Gauss devised for determining the terrestrial field.[2] The value to be measured is deduced from the deflection which the direction of the horizontal component experiences when it is combined with an auxiliary component of known value, also horizontal, but at right angles to it. To determine the direction of the resultant a *magnetometer* is used. The essential part of this instrument is a well-damped small system of magnets suspended by a vertical fibre, as free from torsion as possible (quartz fibres are best), and which can be turned, the azimuth being read off by a mirror.[3] The above-mentioned auxiliary

[1] F. Kohlrausch, *Leitfaden der prakt. Physik.* 7th edition, p. 255 *et sqq.* Leipzig, 1892.

[2] Gauss, *Intensitas vis magnet. terrestris ad mensuram absolutam revocata. Werke*, vol. 5, p. 89; 2 Reprint. Göttingen, 1877. See also F. Kohlrausch, *loc. cit.* pp. 230–236.

[3] The following are the chief points in reference to the construction of a magnetometer, of which there are many types, simple and complicated. The system of magnets must be small, so that the auxiliary component in the space occupied by it is sufficiently uniform, and its moment of inertia is small; on the other hand, its magnetic moment must be as great as possible. Perhaps the best is a thin aluminium disc on both sides of which small mag-

component is produced by means of the action at a distance of an auxiliary magnet, as will be described in the following paragraphs. The magnetic moment of this magnet may either be known at the outset, or its product into the horizontal intensity, which is the quantity to be measured, may be determined by one of the following methods :—

A. *Observation of Oscillations.*—The auxiliary magnet, whose (unknown) permanent magnetic moment may be $\mathfrak{M}$, is suspended horizontally in the place where the horizontal component $\mathfrak{H}$ is to be determined. In this position its period τ, or its frequency $N = 1/\tau$, is observed, from which is obtained [by equation (8), § 23]

$$(1) \qquad \mathfrak{M}\,\mathfrak{H} = \frac{4\,\pi^2 K}{\tau^2} = 4\,\pi^2 N^2 K$$

The moment of inertia K of the auxiliary magnet may be either calculated, or else determined experimentally, by some dynamical method.

B. *Method of Weighing.*[1]—The auxiliary magnet is fastened vertically in the middle of the scale-beam. Let the balance swing in the magnetic meridian, and let the difference in weight corresponding to half a turn about the vertical be δM. If D is the length of the scale-beam, g the acceleration of gravity, we obtain

$$(2) \qquad \mathfrak{M}\,\mathfrak{H} = \frac{1}{2}\,g\,\delta M D$$

C. *Bifilar Suspension; Torsion.*—In the arrangements previously described, the auxiliary magnet oscillates about a position of equilibrium—that is, the horizontal (A) or the vertical direction (B) in the magnetic meridian. In the former case the horizontal component, and in the latter the vertical component, obviously induces in it a certain magnetisation, which is super-

nets are fixed. As electromagnet copper dampers are of little use in such feeble systems, air-damping, which can be regulated, is to be preferred. Although torsion may nearly always be neglected when quartz fibres are used, a torsion head should in all cases be fitted. It is, lastly, very convenient, if the mirror can be turned in reference to the system, and if the case, which must not contain the least iron, is arranged for being set upon a horizontal plane, or else suspended against a wall.

[1] Toepler, *Wied. Ann.* vol. 21, p. 158, 1884.

posed on the existing permanent magnetisation, and therefore increases the moment $\mathfrak{M}$. The correction due to this is, however, only small in observations on terrestrial magnetism; nevertheless, it is always better to place the auxiliary magnet in a nearly east and west position. The transverse magnetisation then produced has scarcely any effect. For instance, the auxiliary magnet may be suspended in this position to a bifilar or torsion arrangement, the directive torque of which is known (note 3, p. 297). The latter is multiplied with the tangent of half the deflection produced by reversing the magnet to the west and east direction. The product gives then directly the value $\mathfrak{M}\,\mathfrak{H}$.

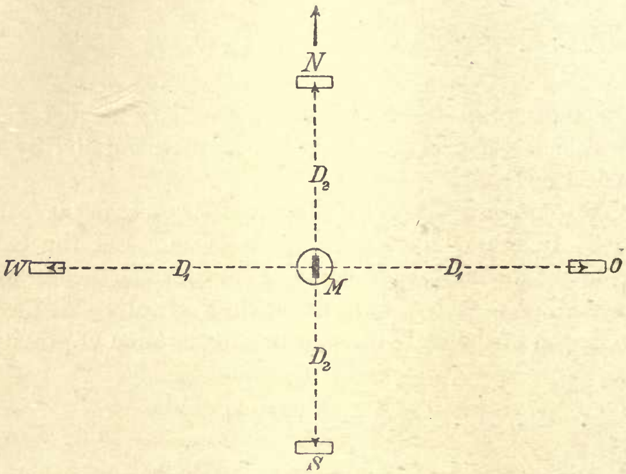

FIG. 61

§ 191. **Observations of Deflection.**—After the auxiliary magnet has been taken away, the magnetometer M is put in its place. Its magnetic system then sets itself in the direction of the field, which we have called the magnetic meridian, and which is marked NS in fig. 61. In order now to produce a deflection of the magnetic system, the auxiliary magnet is placed in one of two different positions, its own direction in both cases being either west-easterly or east-westerly.

1. *First Principal Position.*—Auxiliary magnet at distance D_1 in W or O at right angles to the meridian. The component $\mathfrak{H}_1$ induced by it at M is also at right angles to the meridian, and amounts to [equation (5), § 22]

$$(3_1) \quad . \qquad \mathfrak{H}_1 = \frac{2\,\mathfrak{M}}{D_1^3}\left[1 + \frac{1}{2}\frac{L^2}{D_1^2} + \frac{3}{16}\frac{L^4}{D_1^4} + \cdots\right]$$

in which L is the geometrical (or 'virtual,' § 210) length of the auxiliary magnet.

2. *Second Principal Position.*—Auxiliary magnet at N or S at distance D_2, at right angles to the meridian, as well as the component $\mathfrak{H}_2$ due to it, which in this case has the following value [equation (6), § 22][1]:

$$(3_2) \quad . \qquad \mathfrak{H}_2 = \frac{\mathfrak{M}}{D_2^3}\left[1 - \frac{3}{8}\frac{L^2}{D_2^2} + \frac{15}{128}\frac{L^4}{D_2^4} - \cdots\right]$$

The deflections α_1 or α_2 due to the second component are obviously given by the following equations:

$$(4) \quad . \quad . \qquad \tan\alpha_1 = \frac{\mathfrak{H}_1}{\mathfrak{H}} \qquad \tan\alpha_2 = \frac{\mathfrak{H}_2}{\mathfrak{H}}$$

From equations (3_1) or (3_2) and (4), neglecting the factors in the brackets, we get, as a first approximation,

$$(5) \quad . \qquad \frac{\mathfrak{M}}{\mathfrak{H}} = \tfrac{1}{2}\,D_1^3 \tan\alpha_1 \quad \text{or} \quad \frac{\mathfrak{M}}{\mathfrak{H}} = D_2^3 \tan\alpha_2$$

After having expressed $\mathfrak{M}\,\mathfrak{H}$ and $\mathfrak{M}/\mathfrak{H}$ as functions of determinate quantities, not only the intensity $\mathfrak{H}$, but also incidentally the moment $\mathfrak{M}$ of the auxiliary magnet, may be calculated. If the latter is already known, $\mathfrak{H}$ is obviously obtained from equation (5) without further determinations.

These simplified equations, however, only hold for distances which are very great in comparison with the length of the deflecting magnet (§ 22); but as the deflections obtained at such distances are usually too small, the magnet must be brought nearer the magnetometer, so that several members of the series in equation (3_1) or (3_2) have to be brought into account. The length of the magnet is, however, usually eliminated by observing at two successive distances (Kohlrausch, *loc. cit.*, p. 231).

Gauss's method is used mostly for determining the absolute value of the earth's horizontal intensity. It may, however, in principle be applied to measuring the horizontal components of

[1] An elementary deduction of equations (3_1) and (3_2) is found, for instance, in F. Kohlrausch, *loc. cit.* pp. 389, 390.

uniform fields due to any agent, provided their intensity does not exceed 1 C.G.S. If the field is stronger, it is scarcely possible to produce sufficient deflection by means of magnets of the usual size.

B. *Electrodynamic Methods*

§ 192. **Measurement of a Dynamical Force.**—It was mentioned in § 1 that the magnetic field can be completely defined by the two chief forms in which it manifests itself—the electrodynamic and the inductive; and for the practical methods of measurement based on this, reference was made to the present chapter.

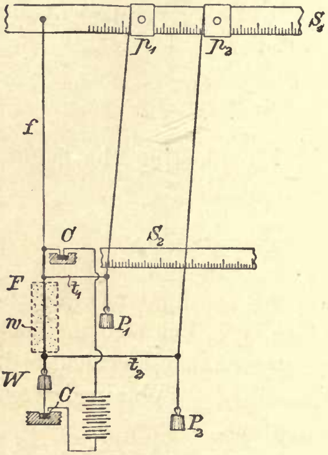

FIG. 62

As regards electrodynamic[1] methods, we will mention the following simple arrangement, due to Lord Kelvin.[2] In the field F a metal wire hangs between two pole-pieces, supposed horizontal and at right angles to the plane of fig. 62. By means of two mercury cups C, C a current of known strength I (in decamperes) is passed. If then the intensity of the field is $\mathfrak{H}$, its effective height L, and its direction is such that a force, $\mathfrak{F}$, expressed in dynes, is exerted, say, towards the left on the wire, this, from electro-dynamical principles, is given by the equation

$$\mathfrak{F} = I \mathfrak{H} L \qquad (6)$$

This force is held in equilibrium by the tension of the threads t_1 and t_2, which are fastened to the string pendulums $\overline{p_1 P_1}$ or $\overline{p_2 P_2}$. It needs no explanation how that force can be

[1] By electrodynamic action all forces are understood which are exerted on conductors conveying currents in the magnetic field, no matter whether the field is due to other conductors, or depends on other sources, such as rigid magnets.

[2] A. Gray, *Absolute Measurements in Electricity and Magnetism*, vol. 2, p. 701, London, 1893.

determined in absolute measure from the readings on the scales S_1 and S_2, as well as from the weights P_1 and P_2.

The wire conveying the current may also be stretched. By the action of the electrodynamic forces it will then sag laterally, like a tight string. The elongation, which may be measured by a microscope, or by mirror reading, is, as a first approximation, proportional to the force, and therefore, with a constant current, proportional also to the intensity of the field. This principle has been applied in Ewing's magnetic curve-tracer (§ 214). The method described is suitable for fields of the order of 100 C.G.S. units and upwards.

§ 193. **Measurement of a Torque.**—The simple method which has been described is obviously not very accurate. More trustworthy results are obtained when the conductor is bent into a loop of one or more turns; and the couple exerted in general on such a loop—that is to say, on a coil in the field—is determined by one of the known methods, such as weighing, torsion, or bifilar suspension.

Stenger has described a special apparatus based on this last principle,[1] which may be regarded as an inversion of the well-known 'syphon recorder' of Lord Kelvin, and of the bifilar galvanometer of F. Kohlrausch.[2] A small coil is suspended by two wires, which at the same time convey the current I, which is separately measured. Let the plane of the coil be parallel to the field; let the total area of the windings be S, the directive torque[3] of the bifilar D, and the deflection observed be a. The field-intensity to be measured is then given by the equation

$$(7) \qquad \mathfrak{H} = \frac{D \tan a}{S I}$$

Stenger thus succeeded in determining in absolute measure fields of the order of 100 C.G.S. to within 0·1 per cent. with certainty and convenience. The sensitiveness of such methods may be lowered at will by diminishing the current through the

[1] Stenger, *Wied. Ann.* vol. **33**, p. 312, 1888. Compare also Himstedt, *Wied. Ann.* vol. **11**, p. 829, 1880.

[2] F. Kohlrausch, *Wied. Ann.* vol. **17**, p. 752, 1882.

[3] In a suspended system directed by any appliance we suppose the directive torque referred to unit deflection expressed in circular measure (57·296°).

coil, and they may thus also be used for measuring the most intense fields.

Torsion, applied as directive torque, has been used in measurements of fields, which A. du Bois-Reymond[1] has published. The arrangement in this case may be regarded as an inversion of the Deprez-d'Arsonval galvanometer.[2] On a similar principle Edser and Stansfield[3] have recently constructed a convenient portable instrument for measuring a field. A plate of mica supports a coil of thin copper wire, which is stretched by two German silver wires conveying the current. One of the wires is fastened to a torsion head, which at the same time is ingeniously made to serve as commutator. A Hellesen dry cell furnishes a nearly constant current of, at most, two centiamperes, since the coil itself has a resistance of 50 ohms. By inserting independent resistances the sensitiveness can be diminished, if necessary. The instrument, it is stated, measures fields of any direction from 1 C.G.S. upwards, with an accuracy of about two per cent. It is especially suitable for measuring leakage in and about dynamo machines, as Edser and Stansfield show (*loc. cit.*) by some examples.

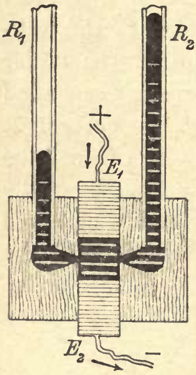

Fig. 63

§ 194. **Measurement of a Hydrostatic Pressure.**—The method of measuring force just mentioned may, lastly, be so modified that, instead of a solid wire conductor, a liquid one—mercury—is used. This is contained in a flat insulating cavity perpendicular to the direction of the field, which is again assumed to be horizontal, and at right angles to the plane of the figure. The width of the cavity d does not exceed a fraction of a millimetre. Let the current traverse the mercury in a vertical direction, for which purpose two platinum electrodes, E_1 and E_2 (fig. 63), are in contact with it. By electrodynamic action a lateral thrust is exerted on the mercury, driving it out, so that a difference of level results between the tubes R_1 and R_2, which represents a pressure P

[1] A. du Bois-Reymond, *Elektrotech. Zeitschrift*, vol. **12**, p. 305, 1891.

[2] Deprez and d'Arsonval, *Compt. Rend.* vol. **94**, p. 1347, 1882.

[3] Edser and Stansfield, *Phil. Mag.* [5], vol. **34**, p. 186, 1892.

equivalent to that thrust. It may be readily shown [1] that the condition of equilibrium is represented by the following equation:

$$\mathfrak{H} I = P d \qquad (8)$$

It will thus be seen that, *ceteris paribus*, the pressure, assumed to be expressed in the equation in dynes per sq. cm., is inversely proportional to the clear width of the cavity, and accordingly the latter is taken as small as possible.

Lippmann [2] has devised a galvanometer, depending on this principle, in which I in the equation is the quantity to be measured. By reversing this an arrangement is obtained by which $\mathfrak{H}$ may be measured, as in the previous cases. An apparatus of this kind was first described by Leduc.[3] The mercury is contained between two glass plates, which are kept apart by pieces of microscope glass, and cemented by Canada balsam. The tubes R_1 and R_2 were widened in one place, in which were the surfaces separating the mercury and the supernatant water or alcohol. In this way the sensitiveness could be considerably increased, as also by inclining one of the tubes—at the cost, however, of the accuracy of the measurement.

Leduc's instrument was afterwards somewhat modified by the author,[4] in order to adapt it for use in the narrow spaces between the pole-pieces of powerful electromagnets. Neither the galvanometer nor the portable instrument for measuring the field is suitable for absolute measurements, from the difficulty of accurately determining the thickness of the layer of mercury; but its use for the relative determination of horizontal fields of small extent, of the order of 1000 C.G.S., with an accuracy of 0·5 per cent., is so much the more convenient. By using water or alcohol for measuring the pressure, the apparatus is adapted also for a lower order of magnitude of, say, 100 C.G.S.

[1] Mascart and Joubert, *Electricity and Magnetism*, English edition, vol. **2**, p. 243.

[2] Lippmann, *Compt. Rend.* vol. **98**, p. 1256, 1884. *Journal de Physique* [2], vol. **3**, p. 384, 1884.

[3] Leduc, *Journal de Physique* [2], vol. **6**, p. 184, 1887.

[4] du Bois, *Wied. Ann.* vol. **35**, p. 142, 1888. See also Field and Walker, *The Electrician*, vol. **32**, p. 186, 1893.

C. *Methods of Induction*

§ 195. **Arrangement of the Exploring Coil.**—After describing in the previous section the methods of measurement based on electrodynamic actions, we now pass to the second chief manifestation of the magnetic condition, the inductive action, wherein we shall provisionally confine ourselves to indifferent media.

In the introduction (§§ 2, 4) the most important properties of the magnetic field were already elucidated by their aid, experiments with an exploring coil having then been described, only the outlines of which could be explained. After, in § 189, having pointed out the use of exploring coils for determining the distribution of a field, we must now describe the mode of carrying out such experiments, in so far as measuring the absolute value of intensity is concerned. Let S be the area of the exploring coil, R the total resistance of the secondary circuit, Q the quantity of electricity conveyed on the sudden production or cessation of the field $\mathfrak{H}$, which is assumed to be at right angles to the exploring coil. We have shown [equation (1), § 4, and § 64] that then

$$(9) \quad . \quad . \quad . \qquad \mathfrak{H} = \frac{QR}{S}$$

In making such experiments it must be remembered that by means of the exploring coil we can only detect or measure variations in the flux of induction—which in the indifferent air-space is identical with ($\mathfrak{H}\,S$)—and cannot determine its value at a given instant. Accordingly, one or other of the following methods is used:—

1. The exploring coil is placed in a fixed position, which is that of the direction of maximum induction. The field is then suddenly produced, stopped, or its direction reversed. If it is exclusively due to a current, this is attained by making, breaking, or reversing.

2. The exploring coil is rapidly removed to a position in which the intensity of the field may be neglected. For this purpose it is usually fixed to a long lever, which springs back at the desired moment, by means of a spring, for example.

3. The exploring coil is rapidly turned, so that the flux of

induction traverses it in the opposite direction. This method is to be recommended, provided the space is sufficient, for it is easily effected, and, moreover, twice the quantity of electricity is set in motion. The exploring coil is fastened to a handle, which can be turned about an axis in the plane of the coil.

The area S of the windings can be determined by various methods :—[1]

1. By directly measuring the diameter of each layer of windings with the kathetometer or the dividing machine; or measuring the perimeter by a tape, allowing for the thickness of the wire.

2. By measuring the length of wire used in coiling. The formulæ required for this are given by F. Kohlrausch (*loc. cit.*).

3. By the magnetic action at a distance of the coil conveying the current. Its magnetic moment is $\mathfrak{M} = IS$ (§ 6), and can be measured magnetometrically (§ 209). If, further, the same current is passed through a tangent galvanometer, its value may be completely eliminated.

The *ballistic* method may be used for measuring fields of any intensity. The sensitiveness of the galvanometer, the area and the number of turns of the exploring coil, and the resistance of the secondary circuit are factors by which, when suitably fixed, the sensitiveness of the method may be raised or lowered to almost any extent. This general applicability is a chief advantage of the ballistic method.

§ 196. **Ballistic Galvanometer.**—The quantity of induced electricity Q set in motion is almost always determined by the ballistic galvanometer, whence the usual name is derived. In selecting a galvanometer the chief point to be attended to is that its period can be made as long as possible in comparison with the time required by the variation of the flux of induction. A galvanometer which swings too rapidly may often be utilised for ballistic purposes by fixing small light bodies to its magnetic (for instance, a bit of match fixed horizontally, or an aluminium stirrup often provided with the galvanometer), so that without any considerable increase of load the moment of inertia, and

[1] F. Kohlrausch, *Leitfaden*, 7th ed. p. 343; Heydweiller, *loc. cit.* §§ 152–155; Himstedt, *Wied. Ann.* vol. **26**, p. 555, 1885.

therewith the period, are greater. The efficiency of a galvanometer is best characterised by the following quantities [1]:—

The *current sensitiveness* S_s is the deflection in scale-divisions per microampere, if the distance of the scale is 2000 parts, and the period is 10″.

The *ballistic sensitiveness* S_b is the deflection in scale-divisions of the scale per microcoulomb, if the distance of the scale is 2000 parts, and the period 10″.

In order to eliminate the resistance R_g of the galvanometer, the *normal sensitiveness* $\mathfrak{S}$ then is introduced, which is defined by equations

$$\mathfrak{S}_s = S_s/\sqrt{R_g}$$

or

$$\mathfrak{S}_b = S_b/\sqrt{R_g}$$

Taking the normal period of 10″, and assuming that damping may be neglected, the two conditions of sensitiveness are connected by the equation

$$(10) \quad . \quad . \quad . \qquad \mathfrak{S}_b = \frac{2\pi}{10}\mathfrak{S}_s$$

Now in ballistic experiments as those it is unnecessary that the damping be excessively small; it is, rather, far more convenient to work with as strong damping as possible, the extent of which is limited by one condition only; the deflection must always remain proportional to the quantity of electricity passing through the galvanometer, a condition which can easily be checked by suitable calibration. Owing to damping the ballistic sensitiveness decreases somewhat; if in these conditions we denote it by $\mathfrak{S}_b'$, and if $\mathfrak{m}$ is the 'damping-ratio,' then, according to Ayrton, Mather and Sumpner,[2]

$$(11) \qquad \mathfrak{S}_b = \mathfrak{S}_b'\,[1 + 0{\cdot}500\,(\mathfrak{m} - 1) - 0{\cdot}277\,(\mathfrak{m} - 1)^2 + 0{\cdot}130\,(\mathfrak{m} - 1)^3]$$

[1] Compare du Bois and Rubens, modified astatic galvanometer, *Wied. Ann.* vol. **48**, p. 248, 1893.

[2] Ayrton, Mather, and Sumpner, *Phil. Mag.* vol. **30**, p. 69, 1890. The damping-ratio is the ratio between the amplitudes of two immediate successive half-swings.

Besides the calculation we have given for deducing $\mathfrak{S}_b'$ from the sensitiveness to steady currents, this quantity may also be determined experimentally. A condenser of known capacity is charged to a given potential, and the quantity of electricity defined thereby is directly discharged through the galvanometer. This method is, however, not advisable in practice; it is not usual to measure the absolute sensitiveness of the galvanometer, but to graduate this in the given secondary circuit, directly as it were, with regard to flux of induction. That is to say, a factor is determined which, multiplied into the deflection, gives at once the product QR; in other words, the corresponding flux of induction. If, further, this is divided by the known area of the exploring coil, the value of the intensity is obtained. For this purpose a convenient flux of induction is required, the absolute value of which is known, and which can at any time be reproduced; that is a

§ 197. **Standard Flux of Induction.**—Various arrangements may be used for this purpose; all of them have a secondary coil,

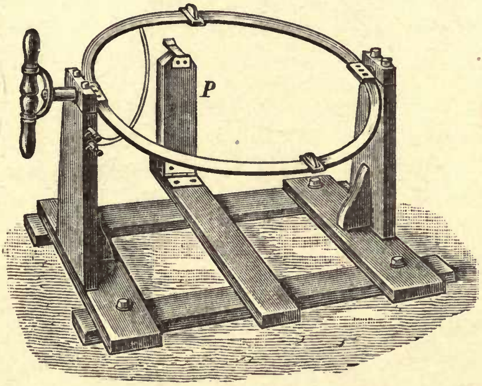

FIG. 64

which is inserted in the secondary circuit containing the ballistic galvanometer to be standardised.

In the first place, we must mention W. Weber's earth-inductor (fig. 64). By means of a handle the circular coil is moved rapidly from the horizontal (or vertical) position through an angle of 180°, against a suitable stop. The total variation

of the flux of induction to be taken into account is equal to twice the product of the area of the coil into the horizontal (or vertical) component of the earth's field; the latter must therefore be first determined, and as it varies considerably with time, especially within a laboratory, this is a circumstance which speaks against the use of the earth-inductor. And the general tendency at present is not to introduce factors involving terrestrial magnetism into measurements wherever this is possible.

According to Lord Kelvin, it is more convenient to use a primary standardising coil without a core, the number of turns of which n, and the geometrical dimensions (length L, area S) are accurately determined, and which conveys a standard current I_A determined in absolute measure. The reversal of this current then corresponds to the following variation of the flux of induction [§ 6, equation (7)].

$$(11) \quad . \quad . \quad . \quad \delta \mathfrak{G} = \frac{8 \pi n S I_A}{L}$$

In § 84 the use of this method for graduating a ballistic galvanometer was explained by reference to an example.

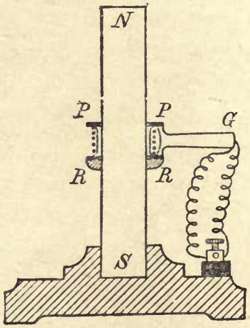

FIG. 65

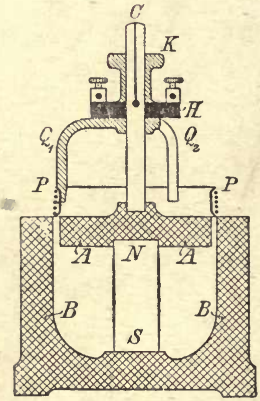

FIG. 66

The permanent flux of induction of as constant a magnet as possible $\overline{NS}$ (fig. 65) may further be used as standard, by letting an exploring coil, rest on a fixed ring RR in the

middle; by means of a handle G, or by means of a suitable spring, this may be rapidly drawn off.

A modification of this simple arrangement has been designed by Hibbert,[1] so as to form a permanent standard field. Fig. 66 represents a vertical section through this apparatus. The magnetic circuit of a good steel magnet $\overline{N\,S}$ is closed by an iron disc $A\,A$, and iron shell BB, with the exception of a narrow annular air-gap. On account of the small demagnetising action and by previous artificial 'ageing' of the steel magnet (§ 150) a permanent flux of induction as constant as possible is obtained.

The exploring coil can be moved closely through the air-gap; for this purpose it is fixed by three radial cross arms, Q_1, Q_2, Q_3, to an ebonite disc, and by means of a knob K, it can be raised or lowered along the rod C. The vertical displacement is accurately limited by means of stops, and therefore corresponds to a definite variation of the flux of induction. The constancy of the standard may be judged from Table IX, in which are given Hibbert's experiments with three of his apparatus, extending over a long period.

The greatest variation in the instruments I and II amounts, as will be seen, to $\frac{1}{2}$ per cent., and with III to 1 per cent. The last measurement denoted by * has been kindly communicated to the author; it shows that within two years there has been no alteration exceeding the errors of observation. According to Hibbert, this constancy depends on the condition that the induction $\mathfrak{B}$ in the steel magnet shall not exceed the value 5,000 C.G.S. He proposes to adjust the flux of induction of the standard to a round number such as 20,000 C.G.S., which would be obtained by suitable stops for the exploring coil; apart from leakage a steel magnet 4 sq. cm. in section would then be necessary to obtain this value with $\mathfrak{B} = 5{,}000$ C.G.S.; owing to the greater section the mean-field intensity in the air-gap is only about the tenth part of the value of the induction in the steel magnet.

[1] Hibbert, *Phil. Mag.* [5], vol. **33**, p. 307, 1892.

TABLE IX

Date	Temperature	Flux of induction $\mathfrak{G}$ I	II	III
April 16, 1891	20°	21 790	—	—
„ 22 „	12·5°	730	—	—
„ 23 „	13·5°	710	32 360	—
May 8 „	16·5°	710	420	—
„ 23 „	13°	680	330	—
„ 30 „	16°	720	410	29 290
June 6 „	18°	720	380	270
„ 12 „	22°	780	470	260
„ 29 „	21·5°	720	345	290
July 10 „	19°	790	510	500
„ 27 „	20°	700	470	550
„ 31 „	17·5°	780	460	530
Sept. 22 „	16°	690	400	470
Nov. 10 „	13°	700	400	480
„ 4, 1893*	—	720*	—	—
Maximum variation		110	180	290
Mean intensity $\mathfrak{H}$ in the air-gap		515	770	700

§ 198. **Measurement of a Field by Damping.**—A method presenting theoretical interest is that of determining the intensity by the electromagnetic damping of a coil oscillating in a field. The damping, as is known, is due to the reaction of the current induced in a closed coil on the field: the method must, therefore, be classed among the induction methods; it has a certain similarity with one of the methods of determining the ohm. The coil has either a bifilar or unifilar suspension in the field, the plane of its winding being parallel to the direction of the field; the period of oscillation τ, or the frequency N, is determined under the influence of the directive force of the suspension alone and with an open circuit. The latter is then closed through an adjustable anti-inductive resistance (a rheostat for example), and this is gradually diminished until the swinging is just dead-beat, a condition which may be ascertained with considerable accuracy. If the corresponding total resistance is R in C.G.S. units (that is in millimicrohms), the moment of inertia of the coil is K, the total area of its windings is S; it may be shown that the intensity of the field is given[1] by

$$\mathfrak{H} S = \sqrt{4 \pi N R K} \qquad (12)$$

[1] A. Gray, *Absolute Measurements*, part ii. vol. 2, p. 708, 1893.

The air-damping is here disregarded, and several factors have been neglected in the course of calculation. This method is, as stated, of theoretical interest rather than practical value; yet in certain circumstances it may do good service in approximately determining or comparing fields of the order of 1,000 C.G.S. In experimental physics, besides the methods of precision which have been worked out in all their details, there is room for simple ones by means of which approximate quantitative determinations may be rapidly made such as the above.

D. *Magneto-optical Methods*

§ 199. **Rotation of the Plane of Polarisation.**—A very convenient method of measuring a field depends on a determination of the magneto-optical rotation of the plane of polarisation of

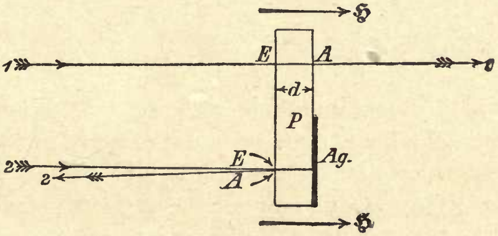

FIG. 67

light in transparent substances, which, as is well known, was discovered by Faraday. Let a plane parallel plate P, of such a substance of thickness d, be placed at right angles to the direction of the field (fig. 67). If then a linearly polarised ray of light 1 · 1 passes parallel to the latter, its plane of polarisation undergoes a rotation ε in the plate, which is proportional to the variation $\Delta\,\Upsilon$ of the magnetic potential (§ 48) between the point at which it enters and that at which it emerges; if we assume the field to be uniform, then evidently $\Delta\,\Upsilon = \mathfrak{H}\,d$, and therefore

$$(13) \qquad \varepsilon = \omega\,\mathfrak{H}\,d$$

The factor of proportionality ω is equal to the rotation per unit variation of magnetic potential; it depends in addition only on the wave-length λ of the light, as well as on the nature

and—to a slight extent only—on the temperature of the substance traversed; it is called *Verdet's constant.* Its sign defines the direction of the rotation with reference to the direction of the field[1] (plain arrows in fig. 67); the direction of the propagation of light (feathered arrows) is immaterial. If, therefore, the plate is partially coated with silver *Ag*, and a pencil of rays 2·2 is allowed to fall on it under a very small angle of incidence, and is reflected as shown, the rotation is doubled; that is, we have

(14) . . . $\varepsilon = 2\,\omega\,\mathfrak{H}\,d$

The numerical value of Verdet's constant has been accurately determined in absolute measure for the two liquids most used, water and bisulphide of carbon; for sodium light ($\lambda = 58{\cdot}9$ microcentimeters); its value is in minutes per unit variation of potential $[\Delta\,\Upsilon]$ at a temperature of 18° is[2]

Water (H_2O)	: + 0·0130′ per $[\Delta\,\Upsilon]$
Bisulphide of carbon (CS_2)	: + 0 0420′ „ $[\Delta\,\Upsilon]$

For several other inorganic and organic liquids (acids, alcohols, ethers, &c.) ω is more or less accurately known, as well as for some tolerably well defined kinds of glass.[3] Hence, in order to have absolute measurements, it is only necessary to determine the thickness of the transparent plane parallel plate used. Liquids are investigated in small glass troughs; the rotation in their glass covers is previously determined while they are empty, and afterwards subtracted.

§ 200. **Standard Glass Plates.**—The magneto-optical method has, among others, the advantage that by the formula $\mathfrak{H} = \varepsilon\,/\,\omega\,d$ the intensity is simply proportional to ε, the measurement of which is as convenient as it is accurate. It is, moreover, not at all necessary to use a substance for which Verdet's constant is

[1] It is usual to define the direction of rotations in respect of the direction perpendicular to their plane by saying, for instance, that the direction of the motion of a clock hand is positive in respect of the direction from the dial to the clock works. This corresponds to the 'right-hand system,' in which the ordinary screw, for instance, may be classed. See Maxwell, *Treatise*, 2nd ed., vol. 1, p. 24.

[2] Lord Rayleigh, *Proc. Roy. Soc.* vol. **37**, p. 146, 1884; Arons, *Wied. Ann.* vol. **24**, p. 182, 1885; Köpsel, *Wied. Ann.* vol. **26**, p. 474, 1885.

[3] Du Bois, *Wied. Ann.* vol. **51**, p. 545, 1894.

known. Instead of a layer of liquid, a glass plate will even be generally preferred, the material of which is indeed less well defined, but as to which we are more certain that it is unalterable. Such a plate is graduated once for all in a field of known intensity, and then represents a convenient standard field which exceeds all others in simplicity, portability, and invariability.

In using glass plates which are exactly, or nearly exactly, plane and parallel with transmitted light, the rays twice reflected internally (fig. 67, 1·1) often exert a disturbing influence, as also, with a silvered glass, do the rays directly reflected from the front surface (fig. 67, 2·2); because, in the first case, they experience a threefold, and in the latter no rotation at all. The author has, therefore, had constructed slightly wedge-shaped standards of the densest, strongly rotating Jena flint glass,[1] in which these disturbing images may be screened near the principal image. They are coated with black paint (cross-shaded in fig. 68), with the exception of a square window of about $0{\cdot}5 \times 1{\cdot}0$ cm.; one half of which, T, is free and transmits the light, while the other R is silvered at the back, and can be used for reflection.[2] The latter method is twice as sensitive, and may in many cases be more conveniently used, provided polariser and analyser are arranged for almost normally reflected light. The rotation may be determined by one of the accurate polarimetric methods, which Lippich has recently greatly improved. Owing to the great dispersion, the source of light must be monochromatic; the best is as bright a sodium burner as possible, several of which have been described. Such a standard plate of about 1 mm. thickness is adapted for measuring fields of the order of 1,000 C.G.S.; the latter is of course inversely proportional to the thickness of plate admissible; that is, to the free long dimension of the field.[3]

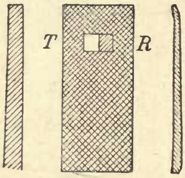

FIG. 68.—½ scale

[1] Du Bois, *Wied. Ann.* vol. **51**, p. 545, 1894.

[2] Within the window the small changes of thickness due to the wedge shape have no detrimental influence.

[3] For further details and examples of the magneto-optical methods of measuring a field, reference may be made to a paper recently published by the author (*Wied. Ann.* vol. **51**, 1894). It may be remarked that it is diffi-

E. *Hall's Phenomenon. Magneto-Electrical Alteration of Resistance*

§ 201. **Hall's Phenomenon.**—We now pass to the discussion of two phenomena produced in metals by the influence of a field, which are suitable for measuring it, although they can scarcely be considered as having been hitherto examined in all directions and explained.

Briefly stated, the phenomenon discovered by Hall consists in the occurrence of a 'rotatory' kind of electrical conduction in metals, which has hitherto been observed only under the influence of a magnetic field. It manifests itself chiefly by the fact that the lines of electric flow no longer run at right angles to the equipotential surfaces, as is always the case in ordinary conditions. Let us consider, for the sake of simplicity, conduction in two dimensions; for instance, a thin strip of metal at right angles to the direction of the field (fig. 69); the lines of flow of the primary current entering or leaving through the electrodes E_1 and E_2 will run, as before, parallel to the free edges; the equipotential lines, however, are deflected from their original orthogonal direction (represented by the plain lines) when the field is excited, into the slanting position (represented by the dotted lines).

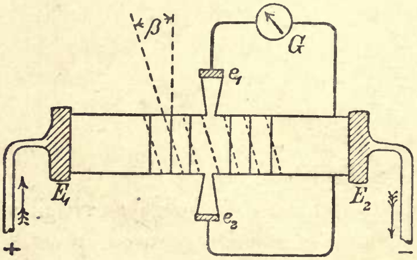

FIG. 69

In places not too near the primary electrodes the rotation is *ceteris paribus* the same in all points, and with metals not ferromagnetic (§ 224) is in general proportional to the field.

cult to prepare thicker glass plates or parallelopipeds so that double refraction shall not exert a disturbing influence. It is necessary, in that case, to have recourse to liquids, notwithstanding their high temperature-coefficient, the formation of striæ, and other disadvantages. Attempts have recently been made to utilise the magneto-optical method also for determining in absolute measure currents passing through coils the dimensions of which have been measured geometrically.

A difference of potential now exists owing to this rotation between points on the edge which before were equipotential; by means of what are called *Hall Electrodes* e_1 and e_2, and of a suitable galvanometer G, this can easily be measured. Now the deflections of the latter are proportional to the intensity of the field, and may theoretically, therefore, be used for measuring it;[1] since, further, they are also proportional to the primary current, by gradating this, the sensitiveness of the method may be regulated; it is, however, specially suited for determining powerful fields; it is possible that samples suitably prepared may serve as standards. This method has not hitherto been practically developed.

§ 202. **Measurement of a Field by Bismuth Spirals.**—The following phenomenon is connected with that described in the last section. The electric resistance of a metallic conductor is in general altered when it is brought into a magnetic field. Without entering upon the details of this phenomenon, it may merely be mentioned that, as Righi first found, the behaviour of bismuth in this respect is very striking; in an

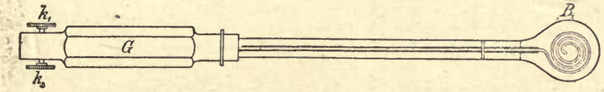

FIG. 70

intense field its resistance may in certain circumstances be more than trebled. Hence, Leduc proposed to use bismuth wire for measuring magnetic fields. To the endeavours of Lenard and Howard we owe a practical method of measurement based on this phenomenon.[2] They succeeded in preparing by pressure a chemically pure bismuth wire of less than 0·5 mm. diameter; this is insulated, wound into a bifilar flat spiral of 5 to 20 mm. diameter having no appreciable self-induction, and then cemented between two plates of mica. The thickness of the

[1] Kundt, *Wied. Ann.* vol. **49**, p. 257, 1893. The proportionality there mentioned has been confirmed in the case of gold and silver for fields up to 22,000 C.G.S.

[2] Leduc, *Journal de Physique* [2], vol. **5**, p. 116, 1886, and vol. **6**, p. 189, 1887; Lenard and Howard, *Elektrotechn. Zeitschr.* vol. **9**, p. 340, 1888; Lenard, *Wied. Ann.* vol. **36**, p. 619, 1890.

whole preparation amounts to less than a millimetre, its resistance is of the order of 10 ohms; both ends of the bismuth wire are soldered to stout copper wires, which pass through a handle of ebonite G and terminate in binding screws k_1 and k_2; this forms at once a suitable fastening and is also convenient for manipulating the flat spiral.

The plane of the spiral is placed at right angles to the direction of the field; the law connecting the increase of resistance with the intensity of the field may be represented by an empirical curve for each spiral which is very nearly the same. Such a one is given in fig. 71; the abscissæ representing the intensity of the field, and the ordinates the ratio

$$\mathfrak{p} = \frac{R'}{R}$$

of the resistance R' in the field to that in ordinary conditions R.[1] For each resistance observed the corresponding intensity of

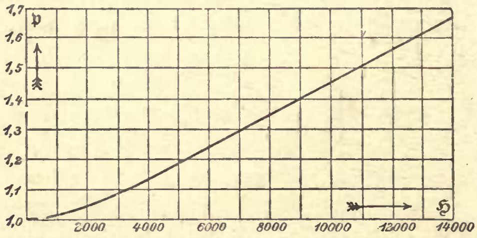

FIG. 71

the field may be read directly on the curve. The measurements of the resistances R' and R, within and without the field, are best made in rapid succession, in order to avoid variations of temperature in the spiral. The latter have of course considerable influence on R' and R, as well as on the ratio $\mathfrak{p}$. According to Lenard, different results are obtained according as the resistance is determined by means of alternating currents—or of electrical oscillations and the telephone, or by means of steady currents.

[1] This curve is taken from a paper by Bruger (*Industries*, vol. 12, May 1893). To save space the range of ordinates $0 < \mathfrak{p} < 1$ is omitted. The curves for various spirals agree to within one or two per cent. Hysteresis has not, so far, been observed in this phenomenon.

The latter method is probably best; Hartmann and Braun have constructed a field-bridge in which the intensity is read off directly from the position of the sliding contact; they have also perfected the construction of the bismuth spirals. For measuring field-intensities of the order of 1,000 C.G.S. the method described offers certain advantages.

[An extended investigation of the behaviour of bismuth-spirals was recently published by Henderson.[1] He finds that the curve of fig. 71 for any given temperature, continues rectilinear up to fields of 40,000 C.G.S. The higher the temperature, the higher, of course, is the initial 'zero-field' resistance, but the less steep the ascent of the resistance-curve, so that it actually meets the curves for lower temperatures at a certain field-intensity. This means that in such a field there is no tempera-

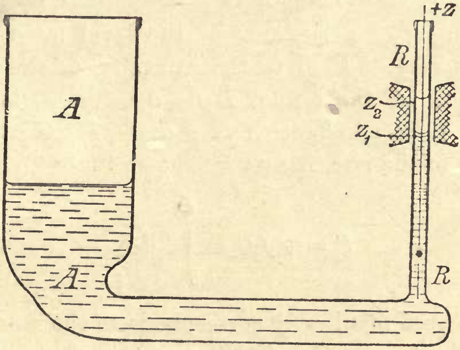

FIG. 72

ture-variation; plotting the resistance as a function of temperature, between 0° and 80°C., Henderson finds a considerable *decrease* for a field of 20,000 C.G.S., practically *no* variation from 10,000 C.G.S., and, of course, the usual increase outside the field. For further details the original paper must be referred to. Quite recently, further light has been thrown on the electric behaviour of bismuth by the work of Sadowsky,[2] and of Dewar and Fleming.[3]—H. du Bois.]

[1] J. B. Henderson, *Wied. Ann.* vol. **53**, p. 912, 1894; *Phil. Mag.* November 1894.

[2] Sadowsky, *Journ. de Physique* (3), vol. **4**, p. 186, 1895.

[3] Dewar and Fleming, *Phil. Mag.* (5), vol. **40**, p. 303, 1895.

F. *Magneto-hydrostatic Method*

§ 203. **Principle of the Method.**—What is called the *magneto-hydrostatic method* was first used by Quincke in an extended investigation of the properties of paramagnetic and diamagnetic (that is in the sense hitherto used indifferent [1]) liquids. If these properties be assumed to be known, or if they are eliminated, the method may conversely be used for determining the intensity of the field.

A *U*-tube, the best form of which is that represented in fig. 72, where the section of the narrow tube $\overline{R\,R}$ may be disregarded in comparison with that of the reservoir $\overline{A\,A}$, contains liquid, the meniscus of which is at first at Z_1. If a magnetic field is produced here, it is observed that the meniscus rises (or falls) according as the liquid is paramagnetic (or diamagnetic) [§ 7]; owing to its large area, the level in $\overline{A\,A}$ does not appreciably vary. If $\mathfrak{H}$ is the intensity of the field, $\mathfrak{I}$ the magnetisation of the liquid, D its density, a its rise due to magnetism, g the acceleration of gravity, P the pressure which corresponds to the rise, it may be shown that [2]

$$(15) \quad . \quad . \quad P = a\,g\,D = \int_0^{\mathfrak{H}} \mathfrak{I}\,d\,\mathfrak{H}$$

This integral manifestly corresponds to the area enclosed by the curve of magnetisation (§ 13), the axis of abscissæ, and the ordinate corresponding to the value of its upper limit (compare also § 148).

In so far as it can be assumed that the liquid has a constant susceptibility κ (compare the author's paper referred to), $\mathfrak{I} = \kappa\,\mathfrak{H}$; if this is introduced into the above integral,[3]

[1] Quincke, *Wied. Ann.* vol. **24**, p. 374, 1885. The method, with some modifications, was afterwards also used for gases and solids.

[2] Compare du Bois, *Wied. Ann.* vol. **35**, p. 146, 1888.

[3] Strictly speaking, κ is the difference between the susceptibilities of the liquid and of the gas above it. Equation (16) was deduced by Quincke (*loc. cit.*) from Maxwell's general formula for electromagnetic stress. The confirmation of that equation by Quincke and others forms, therefore, in any case a support for Maxwell's theory, even though it only refers to feebly magnetic substances (p. 166).

$$(16) \quad . \quad . \qquad P = a\,g\,D = \frac{\kappa}{2}\,\mathfrak{H}^2$$

The order of magnitude of these actions is such that, with a field of 40,000 C.G.S., which for the present can scarcely be exceeded (§ 175), a (diamagnetic) water meniscus would sink by about 0·5 cm., while a concentrated (paramagnetic) aqueous solution of iron chloride would rise by about 100 times as much—that is, by 50 cm.

§ 204. **Practical Execution.**—This action may still further be increased by inclining the tube $\overline{RR}$ at an angle α with the

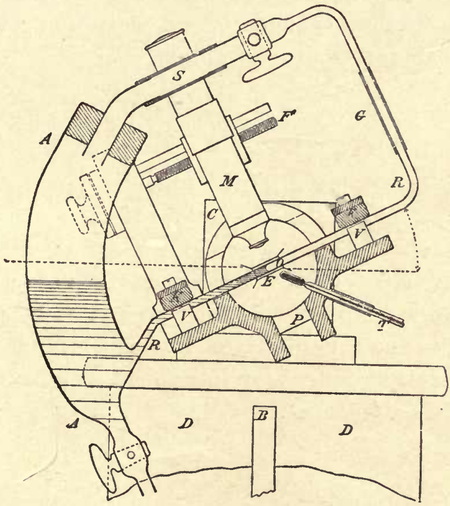

FIG. 73

horizon; a displacement of the meniscus b is then obtained which is connected with the vertical ascent a by the equation

$$a = b \sin \alpha$$

Fig. 73 represents an apparatus constructed by the author (*loc. cit.*); instead of Quincke's U-tube of fig 72, there is a glass vessel $\overline{AARRGS}$ by means of which the field between the pole-pieces of an electromagnet can be investigated at any inclination of the measuring tube, since the whole appa-

ratus can turn about the axis of the field which meets the plane of the figure in E. The displacement of the meniscus is read off by the microscope M fixed to the micrometer screw F.

Powerfully paramagnetic concentrated solutions of iron, or manganese salts, are less suited for measurement, as their viscosity is too great, and in time they are apt to undergo chemical change. For practical purposes, one of the best is a semi-concentrated solution of the green rhombic nickel sulphate ($NiSO_4$, $7\,H_2O$), the density and susceptibility of which are determined once for all in a known field. These do not alter if the liquid is kept in a closed vessel so as to prevent evaporation, and consequent changes of concentration. The intensity of the field is found from the formula

$$(17) \quad . \quad . \qquad \mathfrak{H} = \left(\sqrt{\frac{2\,g\,D}{\kappa}}\right)\sqrt{a}$$

in which the expression within the bracket may be conveniently brought to a round number, 10,000 for instance (if a is expressed in centimetres). The measurements must always be made at about the ordinary equable temperatures of rooms; for further details the work cited must be referred to; since the ascent is proportional to the square of the intensity, the method is only adapted for determining very intense fields, the order of, say, 10,000 C.G.S. for instance.

CHAPTER XI

EXPERIMENTAL DETERMINATION OF MAGNETISATION OR OF INDUCTION

§ 205. **General Remarks.**—Theoretically, all the methods of determining the intensity discussed in the previous chapter may also with more or less considerable alterations be adapted for the absolute or relative measurement of magnetisation; we shall therefore once more consider them in the same order from this point of view, which, however, in practice is quite different. And here, again, we lay chief weight on newer methods, especially in so far as the principles which hold for magnetic circuits apply to them; however, for the sake of completeness the older methods will be briefly mentioned. In making a selection among the many possible methods, we must, as in the determination of field-intensity, be guided by the conditions of the most varied kind, which apply in each case. The object of measurement will in most cases be that of obtaining the normal curve of magnetisation or that of induction. In case the measuring apparatus used does not directly give these curves (§§ 212, 214), their abscissæ $\mathfrak{H}$ are, in the first place, to be determined experimentally by the methods of the previous chapter—or in many cases calculated—and, in the second place, the ordinate $\mathfrak{I}$ or $\mathfrak{B}$ is to be measured. The observations sometimes give directly the former vector—frequently as the magnetic moment per unit volume—and sometimes the latter. When two of the principal vectors $\mathfrak{H}$, $\mathfrak{B}$ and $\mathfrak{I}$ are known, the third may be at once deduced by means of the fundamental expression [equation (13), § 11]

$$\mathfrak{B} = 4\pi\mathfrak{I} + \mathfrak{H}$$

The calculation is particularly simple in the earlier stages of the magnetisation, where we may write $\mathfrak{B} = 4\pi\mathfrak{I}$ (§§ 11, 59).

It is of great importance to allow for the influence of the shape of the body investigated, since this, as has already been sufficiently dwelt on, in many cases far exceeds the influence of the special properties of the metal, the temperature, and other factors; and even an error in numerically eliminating this factor might partially conceal the properties of the ferromagnetic substance with which we are primarily concerned. This fact cannot be sufficiently insisted on, especially as until quite recently it has been frequently overlooked, and this omission has led to a series of errors and false conclusions.

§ 206. **Discussion of the Shape of the Test-piece.**—That omission, however, is strictly speaking only admissible if the shape of the body to be investigated is that of a closed ring, which is also advisable for theoretical reasons. Such a ring should never be forged from a bar, for even the most skilfully made weld always acts more or less like a joint—that is, it has a demagnetising and leaking action (§ 151); the ring should rather be turned from a plate. This has, of course, the disadvantage that the direction of the laminæ or fibres is everywhere parallel, and therefore, in general, not peripheral, the homogeneity too is generally doubtful. A closed ring, finally, is only available for measurements by the ballistic method, against the use of which objections may in many cases be raised (§§ 216, 220), especially when solid undivided material is used.

As regards the use of non-endless figures, the ovoid comes first, because this alone acquires uniform magnetisation in a uniform field. For accurate investigations more especially it is desirable to use elongated ellipsoids ($m \geqq 50, N \leqq 0{\cdot}0181$, Table I, p. 41.) The material used should be as homogeneous as possible; the turning of an ovoid in a lathe offers no great difficulty when using a standard gauge; the volume may be determined by the weight of water displaced, and affords a control of the direct measurement of the dimensions and the excentricity calculated from it. By using the demagnetising-factors of Table I, p. 41 (if necessary interpolated by means of the values of C), the magnetisation curves of the ellipsoid may by shearing (§ 17) be conveniently transformed to normal curves.

If commercial wire is to be used directly for tests, the mean

demagnetising factor for cylinders Table I, p. 41, must be taken into account.[1]

With a magnetically hard material the dimensional ratio of the dimensions may be taken lower than with a softer ferromagnetic substance, because the normal curves of the former are in themselves more distant from the axis of ordinates, and therefore are relatively less influenced by shearing. Apart from special cases, it is scarcely worth while to investigate cylinders whose length is less than twenty times the diameter; if, on the other hand, $\mathfrak{m} > 500$ ($N < 0{\cdot}00018$), the influence of shape may usually be altogether neglected; still a magnetisation $\mathfrak{J} = 1000$ C.G.S. produces a demagnetising intensity $\mathfrak{H}_i = 0{\cdot}18$ C.G.S.

If the demagnetising action of the ends—or, what comes to the same, the magnetic reluctance of the surrounding air—is diminished by a yoke (see § 218), the correction—that is, the amount of shearing of the curves—will be smaller, and somehow or other has to be determined once for all.

§ 207. **Details of the Method.**—The older methods were mostly intended for investigating the moment of permanent magnets; as there are then no magnetising coils, the placing of the magnets is far simpler. Arrangements in which it is suspended by a fibre or hangs to a balance are scarcely applicable when coils have to be used.

In investigating the induced magnetisation of ovoids or cylinders, the magnetising coil must be two or three times as long as the body tested, and must encircle it as closely as possible; the necessity of having a cooling arrangement by means of water between the test-piece and the coil often, however, stands in the way of the last condition. The coil is suitably inserted in a circuit containing a commutator, an apparatus for measuring the current, a storage battery and a variable resistance —which had best be a liquid rheostat—so that the current can be conveniently and gradually reversed, varied, and measured. In many cases a suitable correction has to be made for the magnetic action of the coil, which is of the same kind as that of the test-piece (§ 211).

[1] [This table has since been superseded by the special investigation of Riborg Mann, *Entmagnetisirungsfactoren kreiscylindrischer Stäbe*, Dissertation, August 1895.—H. du Bois.]

According to the kind of curve to be determined, the current is varied by repeated reversals with a series of increasing or decreasing values of the current; or it is varied step by step, which gives the ascending or descending branches of the (statical) hysteresis loop, the area of which may be determined either by graphical integration, or by weighing the corresponding areas cut out of paper in the well-known manner.[1]

It is in general desirable to demagnetise the test-piece as completely as possible before each series of measurements—that is, to destroy the influence of previous magnetic actions. This is best effected by a process which can be described as that of 'diminishing reversals.' The test-piece is subjected to the influence of a series of slowly or rapidly succeeding fields, the intensity of which each turn changes in direction, and at the same time gradually decreases to zero. If the initial value was large enough, the test-piece is afterwards demagnetised. In practice more resistance is gradually brought into the circuit, for which purpose a liquid rheostat is best used while the current reverser is being constantly worked; instead of letting the current decrease, it is simpler, if possible, to draw the test-piece gradually out of the magnetising coil.

§ 208. **Determination of Distribution.**—The distribution of the vector $\mathfrak{J}$ or $\mathfrak{B}$ within a ferromagnetic body is not, of course, accessible to direct observation; this is only possible with phenomena external to the body, from which conclusions as to internal distribution can only be drawn under certain limitations.

In the first place, the normal component of intensity $\mathfrak{H}_\nu$ may be determined as near the body as possible, but outside the surface. This is identical with the normal component $\mathfrak{B}_\nu$, or $\mathfrak{B}'_\nu$, for external or internal points near the surface, but is not identical with the component of intensity $\mathfrak{H}'_\nu$ in the latter points. We have (§§ 52, 56, 58),

$$(1) \qquad \mathfrak{H}_\nu = \mathfrak{B}_\nu = \mathfrak{B}'_\nu = \mathfrak{H}'_\nu + 4\pi\mathfrak{J}'_\nu$$

In most cases the first term on the right may be neglected in comparison with the second (§§ 11, 59), so that we may put $\mathfrak{J}'_\nu = \mathfrak{H}_\nu/4\pi$. The normal component of magnetisation to be hereby determined gives, further, directly the

[1] The dissipation of energy owing to hysteresis may be directly measured by calorimetrical or watt-metrical methods (note, p. 228), upon which we cannot here enter.

strength of the superficial end-elements which define the action of magnets at a distance (§§ 27, 46). In the cases in question no further end-elements occur in the interior (§§ 50, 59).

For the experimental determination of $\mathfrak{H}_\nu$ the following simpler, but not very trustworthy, methods have hitherto been used: (*a*) Repulsion of one end of a long magnetic needle in the torsion balance (Coulomb); (*b*) Oscillations of a short magnetic needle at right angles to the surface in front of the given point of the surface (Coulomb); (*c*) The same, with a soft iron needle; (*d*) Detachment of a small iron ball, or of a suitable piece of iron (Jamin's test-nail). In the last two cases the value $\mathfrak{H}_\nu^2$ is measured. The only method free from objection is to use a small exploring coil, which is laid flat on the surface to be investigated, and is then rapidly moved away.[1] In this way the flux of induction $\mathfrak{H}_\nu S$ through the surface of the windings of the exploring coil is measured; dividing that by S gives $\mathfrak{H}_\nu$.

An arrangement of Rowland's, which is much used in closed or open magnetic circuits with any given centroid but invariable cross-section, is to let the exploring coil surround the magnet, and then move it suddenly backwards, or to pull it away altogether. The momentary current due to the rapid change of position measures the variation of the flux of induction encircled by it, or the number of induction tubes cut (§§ 63, 64). If, for instance, we consider the short cylinder represented (fig. 7, p. 31), and place an exploring coil about the middle, it is manifest that the flux of induction from the surface passed over each time is measured by the stepped motion of the coil. The distribution of iron filings, moreover (§ 189), in the neighbourhood of the surface gives in such cases an approximate idea of the conditions. The investigations communicated in § 88 III and § 92 furnish examples of the application of the method.[2]

A. *Magnetometric Methods*

§ 209. **Plan of the Experiments.**—In discussing Gauss's method (§ 191) it was mentioned that the moment of the auxiliary

[1] This arrangement offers a certain external analogy with the use of 'proof planes' for investigating electrostatic distribution.

[2] See further Mascart and Joubert, *Electricity and Magnetism*, English translation, vol. **1**, §§ 413, 424; vol. **2**, §§ 1211–1218. Also Chrystal, art. 'Magnetism,' *Encyclopædia Britannica*, 9th ed. vol. **15**, p. 242, Edinburgh, 1883.

magnet could be determined at the same time. The method is therefore adapted to determining magnetic moments. In applying it to this purpose it may be considerably simplified by assuming as known the horizontal intensity $\mathfrak{H}$ which directs the magnetometer. It is either determined once for all, or, by means of a local variometer, it is referred to the known value at another place. The instrument in question is adapted for checking the time-variations. The determination of the directive force may also be avoided by referring the moments to be measured to the known moment of a constant permanent magnet; or, better, if a coil of known area (§ 195)—it may even be the magnetising coil used—is traversed by a current determined in absolute value, the latter forms, as it were, a standard of magnetic moment.

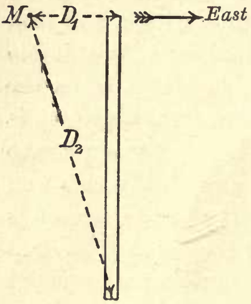

FIG. 74

The magnet, of volume V and cross-section S, whose magnetisation $\mathfrak{J}$ is to be determined, is investigated in the first, or, less conveniently, in the second, chief position (fig. 61, p. 294). We find equation [§ 191, equation (5)]

$$(2) \quad . \quad \mathfrak{J} = \frac{D_1^3}{2V}\mathfrak{H} \tan \alpha_1 \quad \text{or} \quad \mathfrak{J} = \frac{D_2^3}{V}\mathfrak{H} \tan \alpha_2$$

Very long magnets may, according to Ewing,[1] be placed due east or west of the magnetometer (fig. 74) in a vertical position, whereby the top is on about the same level as the latter, that is, so as to give a maximum deflection.[2] It may be shown that the following expression is obtained for the magnetisation :

$$(3) \quad . \quad . \quad . \quad \mathfrak{J} = \frac{D_1^2 \mathfrak{H} \tan \alpha_3}{S\left\{1 - \left(\frac{D_1}{D_2}\right)^3\right\}}$$

[1] The vertical component of the earth's field is best eliminated by means of a special coil; compare Ewing, *Magnetic Induction*, § 40, chapter ii., where all experimental details of the magnetometric method are described.

[2] The use of ovoids with this method, especially with regard to the question of maximum deflection, has quite recently been fully discussed by Naguoka, *Zur Aussenwirkung gleichförmig magnetisirter Rotations-Ellipsoide. Wied. Ann.* vol. 57, 1896; H. du Bois.

in which a_3 is again the observed deflection of the magnetometer, and the signification of D_1 and D_2 is obvious from fig. 74.

§ 210. **Virtual Length of the Magnet.**—The equations given have all been obtained by assuming that the entire action at a distance proceeds from the two ends of the magnet (§ 22). This assumption, however, does not hold good, especially not with short cylinders, and still less with ovoids, the whole surface of which may be considered as overspread with end-elements. It may, however, be shown that the actions at distances, in comparison with whose fourth power that of the length of the magnet may be neglected, may be calculated by supposing the end-elements concentrated in two 'virtual ends,' and with the strength $(\mathfrak{J}\, S)_m$ (§ 19) which prevails in the middle of the magnet. The distance of these ends may be called the *virtual length* of the magnet L'. We have, obviously,

$$(4) \qquad . \quad . \quad . \qquad L' = \frac{\mathfrak{M}}{(\mathfrak{J}\, S)_m}$$

Here $L' < L$, the geometrical length of the magnet, because in the above fraction the denominator—that is, the strength in the middle of the magnet—is always greater than its mean strength $(\overline{\mathfrak{J}\, S})$, which is defined by the equation

$$L\,(\overline{\mathfrak{J}\, S}) = \mathfrak{M}$$

With circular cylinders the ratio of the virtual to the geometrical length depends, strictly speaking, on the magnetisation and on the ratio of the dimensional ratio. We may, with F. Kohlrausch, usually write [1]

$$(5) \qquad . \quad . \quad . \qquad L' = \frac{5}{6} L$$

If this value is introduced instead of L into the series which holds for the first chief position [§ 191, equation (3)], the component of intensity $\mathfrak{H}_1$, due to the magnet at the distance D., is, with sufficient approximation,

$$(6) \qquad . \qquad \mathfrak{H}_1 = \frac{2\,\mathfrak{M}}{D_1^3}\left[1 + \frac{1}{3}\frac{L^2}{D_1^2} + \frac{1}{11}\frac{L^4}{D_1^4} + \dots\right]$$

[1] F. Kohlrausch, *Wied. Ann.* vol. 22, p. 414, 1884. The virtual length L' is often called the ideal or reduced length, or is spoken of as the polar distance.

With ovoids uniformly magnetised, we have in all circumstances, as is capable of strict proof,

$$(7) \quad . \quad . \quad . \qquad L' = \frac{2}{3} L$$

If this again is introduced into equation (3), § 191, we obtain

$$(8) \quad . \qquad \mathfrak{H}_1 = \frac{2\,\mathfrak{M}}{D_1^3}\left[1 + \frac{2}{9}\frac{L^2}{D_1^2} + \frac{1}{27}\frac{L^4}{D_1^4} + \ldots\right]$$

By calculations based on the theory of gravitation a strictly accurate finite expression may moreover be deduced for the action of an ovoid at points at any distance on the prolongation of their axis of rotation; it is as follows:[1]

$$(9) \qquad \mathfrak{H}_1 = \frac{2\,\mathfrak{M}}{D_1^3}\left[\frac{3\,\mathfrak{n}^2}{2\,(\mathfrak{n}^2 - \mathfrak{e}^2)} + \frac{3\,\mathfrak{n}^2}{2\,\mathfrak{e}^2} - \frac{3\,\mathfrak{n}^3}{4\,\mathfrak{e}^3}\log\text{nat}\,\frac{\mathfrak{n}+\mathfrak{e}}{\mathfrak{n}-\mathfrak{e}}\right]$$

$\mathfrak{e}$ here denotes the excentricity of the meridian ellipse (§ 29), $\mathfrak{n}$ the ratio of the distance D, to half the axis of rotation; that is,

$$\mathfrak{e} = \sqrt{1 - \frac{a^2}{c^2}} \qquad \text{and} \qquad \mathfrak{n} = \frac{D_1}{c}$$

If a series is preferred, it may be obtained by dividing and developing the logarithm, and runs as follows:

$$(10) \qquad \mathfrak{H}_1 = \frac{2\,\mathfrak{M}}{D_1^3}\left[1 + \frac{6}{5}\frac{\mathfrak{e}^2}{\mathfrak{n}^2} + \frac{9}{7}\frac{\mathfrak{e}^4}{\mathfrak{n}^4} + \frac{4}{3}\frac{\mathfrak{e}^6}{\mathfrak{n}^6} + \frac{15}{11}\frac{\mathfrak{e}^8}{\mathfrak{n}^8} + \ldots\right]$$

The true value given by this series within the brackets is slightly larger than that which would correspond to the bracketed expression in equation (8).

§ 211. **Von Helmholtz's Method. Compensating Coil.**—If we have three permanent magnets 1, 2, 3, with the virtual lengths L_1', L_2', and L_3', and the moments $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$, the latter, according to Helmholtz, may be determined by hanging the magnets in pairs to a balance, free from iron. Thus, for instance, 1 may be hung vertically at one end, and 2 at the other end parallel

[1] Roessler, *Untersuchungen über die Magnetisirung des Eisens*, pp. 27–31; Inaugural Dissertation, Zürich, 1892. [It is also possible to strictly calculate that action for points not situated on the axis, by the aid of unifocal ovoids; see F. Neumann, *Vorlesungen über Magnetismus*, p. 81. Leipzig, 1885.—H. du Bois.]

to the beam. After reversing one magnet, let the weight required to restore equilibrium be M; if ND is the distance of the knife edges, g the acceleration of gravity, then[1]

$$(11) \quad . \quad (\mathfrak{M}_1\, \mathfrak{M}_2) = \frac{1}{12} \frac{D^4 M g}{1 - \frac{5}{2}\frac{L_1'^2}{D^2} + \frac{10}{3}\frac{L_2'^2}{D^2}}$$

In like manner $(\mathfrak{M}_3\, \mathfrak{M}_1)$ and $(\mathfrak{M}_2\, \mathfrak{M}_3)$ are determined, and then, for instance, we have

$$(12) \quad . \quad \mathfrak{M}_1 = \sqrt{\frac{(\mathfrak{M}_1\, \mathfrak{M}_2)\,(\mathfrak{M}_3\, \mathfrak{M}_1)}{(\mathfrak{M}_2\, \mathfrak{M}_3)}} \quad \&c.$$

We have mentioned the method in this connection, since it also depends on the measurement of actions at a distance, though it can scarcely be called a magnetometric one.

Where permanent magnets are not the subject of investigation, but, as is much more frequently the case, a ferromagnetic substance magnetised by the field of a coil, the action of the coil, and of the conducting wires on the galvanometer, must generally be taken into account. Their action, which is proportional to the current in the coil, may be separately determined and afterwards subtracted from the total action. It is more convenient to neutralise it by a second adjustable *compensating coil*, the action of which is equal but opposite. In coils which are to produce intense fields, such a compensation can scarcely be accurately and permanently obtained, as their action at a distance far exceeds that of the enclosed body. It is in any such case advisable (1) to use a magnetometric system of as small extent as possible, (2) to keep down the temperature of the coils, (3) to twist the two leads together especially before the current reverser, so that they do not enclose any appreciable area. If, however, the compensation is still not complete, the residual differences must be separately determined, and allowed for.

§ 212. **Searle's Curve Tracer.**—By the instrument to be described it is possible to measure simultaneously the intensity

[1] Von Helmholtz, *Berliner Berichte*, p. 405, 1883; Koepsel, *Wied. Ann.* vol. **31**, p. 250, 1887; F. Kohlrausch, *Leitfaden*, 7th ed. p. 248, Leipzig, 1892.

and the magnetic moment induced by it, as well as to trace automatically the corresponding curve of magnetisation.[1] An aluminium fork $\overline{ABDE}$ is suspended by a silk fibre $\overline{KA}$ (fig. 75); a horizontal magnetic needle $\overline{s'n'}$ is fixed to the fork, and tends to set in the magnetic meridian $\overline{SN}$; its motion is damped by a mica vane G'. Between D and E a second silk fibre is stretched which supports a mirror F, on the back of which a vertical magnetic needle $\overline{sn}$ is cemented in the usual manner. As the plane of the mirror is in the meridian, it is evidently only the vertical terrestrial component (and partially also gravity) which tends to set the mirror vertically; the damping of its vibrations is effected by the horizontal mica vane G, which hangs at a small distance above the fixed plate P.

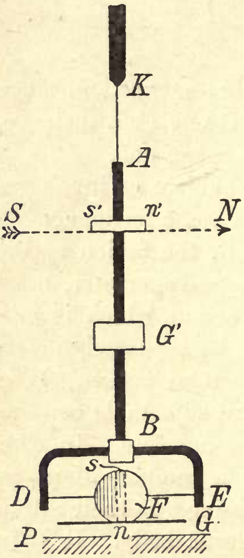

FIG. 75

At a little distance east from F, and therefore above the plane of the figure, the upper end of a vertical ferromagnetic body to be investigated is placed as represented in fig. 74, p. 322; this therefore will produce at F, a component at right angles to the plane of the figure which tends to turn the needle, together with the mirror, about $\overline{DE}$. The deflection affords a measure of the magnetic moment $\mathfrak{M}$ to be determined; the direct action of the magnetising coil is neutralised by the current traversing a second compensating coil placed somewhere west of F; moreover it flows along a third coil which is placed east of the upper needle, $\overline{s'n'}$, and thus imparts to the entire system a deflection about the vertical, which is proportional to the current in the coil, and therefore also to the intensity of the field $\mathfrak{H}$. A spot of light may be reflected from the mirror, the displacement of which in two directions at right angles to each other, as with the well-known Lissajous' figures, is proportional to the values to be measured, $\mathfrak{M}$ (vertical ordinate)

[1] Searle, *Proc. Phil. Soc.*, Cambridge, vol. 7, p. 330, 1892.

or $\mathfrak{H}$ (horizontal abscissa). Hence, the spot of light, as the current increases or decreases, traces the desired curve of magnetisation or hysteresis loop. If the absolute measure of the coordinate scale is specially determined, the indications of this *curve tracer* may also be expressed in absolute measure; according to the inventor's statement, this simple instrument is chiefly adapted for purposes of demonstration.

§ 213. **Eickemeyer's Differential Magnetometer.**[1]—This practical appliance, which is intended for an approximate comparison of the magnetic reluctances of two specimens of iron, has a magnetic circuit, a diagram of which is represented in fig. 76; $s_1 F_1 n_1$ and $s_2 F_2 n_2$ are two equal, heavy, S-shaped blocks of Swedish iron: x and y are the two iron bars to be compared. The flux of induction is in the direction of the arrows $F_2 n_2 x s_1 F_1 n_1 y s_2 F_2$. The coil, a single horizontal turn of which is indicated by C, encircles the two middle cheeks of the iron blocks. A magnetometer needle, not represented in the figure, swings within the vertical field of the coil. This will set vertically, for reasons of symmetry, when the magnetic reluctances x and y are equal; but if x is greater, it will swing to one side, and if y is greater to the other, owing to leakage, which is now no longer symmetrically distributed, and which produces a horizontal component at the point where the needle is. By means of a set of standard bars of a definite kind of iron, arranged like a set of weights, the unknown reluctance x may be approximately equalised in a readily intelligible manner. Besides this zero method, several others, mostly imitations of Wheatstone's bridge, have been described by Edison and others, as to which hardly any information has been published (§ 215). In so far as magnetic shunts are concerned, we may again refer to the discussion § 123.

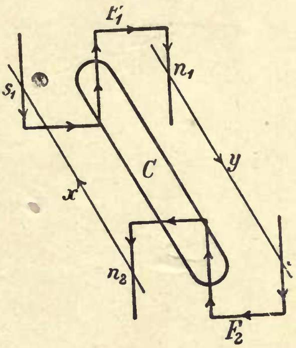

FIG. 76

[1] Steinmetz, *Elektrotechn. Zeitschr.* vol. **12**, p. 381, 1891.

B. *Electrodynamic Methods*

§ 214. **Ewing's Curve Tracer.**—This instrument, which is represented in outline in fig. 77 and in perspective in fig. 78,[1] was invented at the same time as that of Searle, and, like it, has a mirror reading. The perpendicular to the mirror, or, in other words, the reflected ray of light, can again move in two planes at right angles to each other. The deflection in the horizontal plane is proportional to the intensity $\mathfrak{H}$ to be measured; that in the vertical to the induction $\mathfrak{B}$ in the specimen bar. The deflections are, however, produced in quite

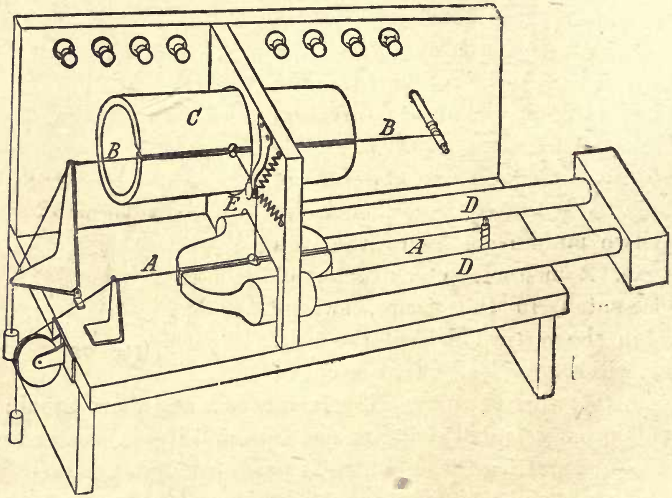

FIG. 77

a different manner, which resembles the method of measurement described in § 192. The wires *AA* and *BB* conveying the current are stretched in the magnetic field by adjustable weights like strings, and when the latter is excited, they experience a sag proportional to the intensity, which is suitably transferred to the mirror *E* by means of cross wire.

Two specimen pieces *DD* of the material are required, which

[1] Ewing, *Elektrotechn. Zeitschr.* vol. **13**, pp. 516, 712, 1892; and vol. **14**, p. 451, 1893. Also Ewing and Klaassen, 'Magnetic Qualities of Iron,' *Phil. Trans.* vol. **184**, p. 985, 1893.

are either solid or are built up of wire, ribbon, or sheet. It is desirable that they should have the rectangular cross-section (about $2{\cdot}4 \times 1{\cdot}2$ cm.) of the two normal specimens, about 45 cm. in length, supplied with the instrument. The specimens are clamped in the two pole-pieces, and likewise in a yoke which connects them at the other end, as seen in fig. 78, where also the coils are represented which magnetise the specimens. The variable current, proportional to the abscissæ, passing through them also passes through the stretched wire $\overline{BB}$. This moves in a constant vertical field, which is produced by the split iron tube C. The horizontal sag of that wire is therefore a measure for $\mathfrak{H}$. The constant current of a few amperes which, in order to magnetise the iron tube, is led through its coil, now also

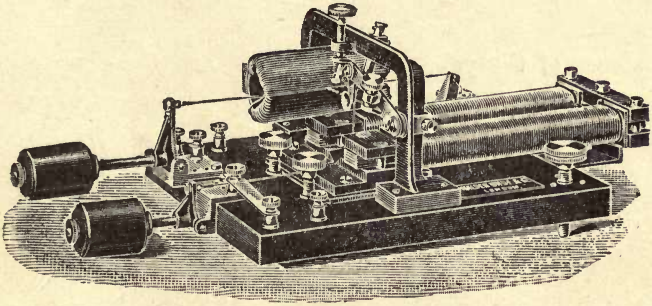

FIG. 78

traverses the wire $\overline{AA}$, which is stretched in the horizontal field in the slit between the pole-pieces of the specimens. Its vertical elongation, therefore measures the intensity of this field—that is, the induction $\mathfrak{B}$ in the specimen bars.

The amplitude of the deflection of the mirror may be regulated both by the strength of the constant current in C and in $\overline{AA'}$, as well as by the lever effect of the weights which stretch the wires. The motion is quite dead-beat. The succession of positions of the projected point of light on the screen is registered photographically or by hand. The scale of co-ordinates may be reduced to absolute measure—for the abscissæ $\mathfrak{H}$ by calculation from the ampere-turns of the magnetising coils, and for the ordinate $\mathfrak{B}$, if necessary, by means of exploring

coils round the specimen bars. Fig. 79 represents curves obtained with such an apparatus. Owing to the magnetic

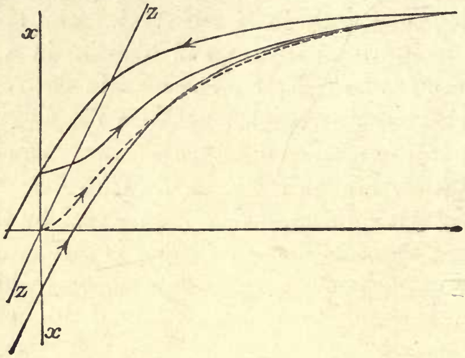

FIG. 79

reluctance of the yoke, the pole-pieces, and the air-gap, these are to be read off from the slanting line $\overline{zz}$, instead of from the axis of ordinates $\overline{xx}$.[1]

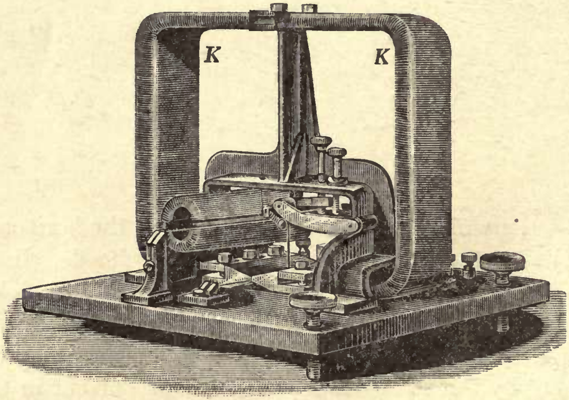

FIG. 80

If the variations of the magnetising current are sufficiently rapid—within about $\frac{1}{10}$ second—the projected spot of light

[1] Ewing has, moreover, described an ingenious kinematic device by which the correction in question is made automatically. The motion of the mirror corresponding to the ordinates is so arranged that the perpendicular to the mirror

appears to trace a continuous line, which then directly represents the induction-curve. The instrument thus forms an apparatus for demonstration as ingenious as it is instructive. The condition above mentioned is satisfied by Ewing by means of a specially constructed rotating liquid rheostat, with or without a commutator.[1] Fig. 80 shows another form of the curve tracer, with a fixed magnetic circuit KK instead of the test bars, as best suited for purposes of demonstration.

§ 215. **Apparatus of Koepsel and of Kennelly.**—The measurement of the field by means of a coil directed by torsion springs is the principle on which is based an apparatus for investigating iron, described by Koepsel.[2] It is represented in fig. 81 in section, and in fig. 82 in plan. The magnetising coils $S\,S$, traversed by a current up to five amperes, produce in the intermediate space a field parallel to their axis. Parallel also to the latter is the plane of the windings of a measuring coil, fastened above and below to a torsion spring, which, at the same time, conveys the current. The auxiliary current through this is adjusted to a constant value—of a deci-ampere, for instance. When the coils SS are excited, the measuring coil will tend to set parallel to them. The angle of torsion which brings it back to zero is a measure of the field of the coil. If a specimen bar is placed in each magnetising coil, the angle of torsion required is far greater; it is, approximately at least, proportional to the induction in the test bars. In the arrangement of the magnetic circuit described, the shape of the specimen is difficult to allow for;

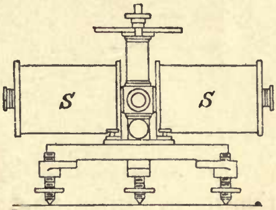

FIG. 81

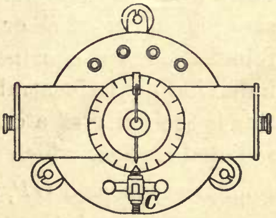

FIG. 82

sweeps a plane inclined to the vertical; hence, when there is no current in the wire, the spot of light does not trace the axis of ordinates, but the line zz, from which then its horizontal deviations take place.

[1] *Electrician*, vol. **30**, p. 65, 1892.

[2] Koepsel, *Verhandl. phys. Gesellsch.*, Berlin, p. 115, 1890; *Elektrotechn. Zeitschrift*, vol. **13**, p. 560, 1892.

Koepsel[1] has therefore recently modified the apparatus in the following manner:—

A magnetising spiral contains a single test-piece, which is enclosed in a yoke (§ 218). The measuring coil is wound on an iron cylinder, which rotates in a suitable cavity in the yoke like an armature. Its deflection is not compensated by torsion, but is read off on a scale by an index. This gives directly the flux of induction in absolute measure; whereby, however, a definite value of the auxiliary current is assumed, and must be separately adjusted or measured.

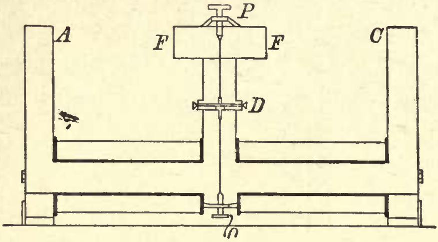

FIG. 83

An arrangement by Kennelly may, in conclusion, be mentioned, by which an approximate comparison of the magnetic reluctances of two specimens of iron is possible, as in the differential magnetometer described in § 213. Both specimens are placed in AF or FC respectively (fig. 83). If the reluctances are equalised, there will, from symmetry, be no induction in the central iron cross-piece FD. The criterion of this is the fixity of a copper disc D, traversed by a current in a radial direction, which can rotate in a suitable slit about the unifilar suspension $\overline{OP}$, which conveys a current.

C. *Induction Methods*

§ 216. **The Ballistic Method**, notwithstanding its many disadvantages, is still undoubtedly the most important for experimental physics—chiefly on account of its universal applicability, and of the unlimited range within which it can be utilised by raising or lowering the sensitiveness. It is less suitable for practical measurements, owing to the considerable disturbance to which it is liable from external influences.

Its application to the measurement of the intensity of a field has been discussed in detail (§§ 195–197), and the way in which

[1] Koepsel, *Elektrotechn. Zeitschr.* vol. **15**, p. 214, 1894.

it is used for determining magnetic distributions has also been pointed out. In Chapter V we have elucidated by an example its application to the investigation of toroids, so that we may here dispense with further general discussion.[1] The present method is the only one which can be used with completely closed magnetic circuits. But as in this case it is not possible to pull away the exploring coil, the flux of induction actually existing at any time can never be ascertained, let alone measured, but its sudden variation can be determined. The ballistic method fails when the time required for such a variation exceeds about a second; in that case it is not possible to make the period of the galvanometer sufficiently long in comparison. With large electromagnets, and with high self-induction, this order of magnitude is very soon reached (§ 170).[2] This forms a chief objection to the ballistic method; in other respects it leaves little to be desired, especially for laboratory purposes.

§ 217. **Isthmus Method.**—It only remains to mention the particular modification in which the ballistic method may be employed for special purposes. For measuring magnetisation at high intensities up to $\mathfrak{H} = 25{,}000$ C.G.S., Ewing and Low[3] introduced what is known as the *Isthmus method*. The ferromagnetic substance to be examined—for example, a piece of iron—is turned on a lathe into the shape of an ordinary bobbin; its ends are either plane, or cylindrical about an axis at right angles to the principal axis. In the former case the iron can be suddenly withdrawn from the magnetic circuit of the electromagnet used in this method; a diagram of the latter form is represented in fig. 84. By means of a handle the whole iron bobbin can be turned about an axis at right angles to the plane

[1] Further details are found in Ewing, *Magnetic Induction*, chapters iii. and iv.

[2] In cores of soft iron thicker than say 1 cm., of high magnetic permeability and electrical conductivity, eddy currents play a considerable part (§ 187). They seem, indeed, to diminish the self-induction, and therefore accelerate variations of current in the magnetising spiral (note, p. 282); but apparently, by their screening action, they produce a considerable time-lag of the vector $\mathfrak{B}$ in respect of $\mathfrak{H}$, the amount of which is difficult to check or bring into calculation.

[3] Ewing and Low, *Proc. Roy. Soc.* vol. **42**, p. 200, 1887; *Phil. Trans.* vol. **180** A, p. 221, 1889.

of the figure, within the pole-pieces in which a corresponding cavity is bored, so that the direction of the field in respect of the bobbin appears suddenly reversed (see fig. 55, p. 260). A secondary coil is closely wound about the narrow neck of the bobbin (the *isthmus*), by means of which therefore the flux of induction through the neck can be ballistically measured, and, on division by the value of the section, gives the induction. At a somewhat greater distance from the neck a second secondary coil is wound, so that between them is a narrow annular, indifferent space. The difference of the ballistic throws observed with each of the secondary coils, divided by the section of the space, is obviously a measure of the field-intensity within this latter;[1] and, from the tangential continuity of the vector in question, this may be regarded as identical with the intensity $\mathfrak{H}$ in the neck itself. In this way both the ordinates of the induction curves and also their abscissæ are determined.

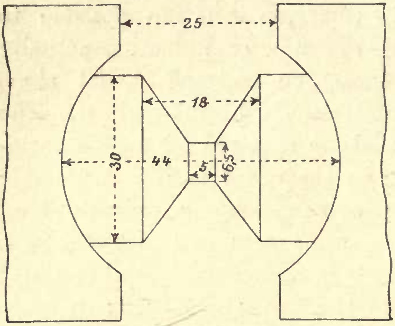

FIG. 84

The latter can be increased far more than is possible with ordinary coils; by rightly shaping the conical part of the iron bobbin, and by using a powerful electromagnet, both the intensity and the uniformity of $\mathfrak{H}$, and therefore also of $\mathfrak{B}$, within the neck are most favourably influenced, as discussed before (§§ 174, 175). With such a coil of annealed Lowmoor wrought iron, in which the section of the neck was 1,500 times less than that of the pole-pieces, Ewing and Low obtained the following values for the magnetic vectors:

$$\mathfrak{H} = 24{,}500, \quad \mathfrak{B} = 45{,}350, \quad \mathfrak{J} = 1660$$

[1] If the two secondary coils—whose number of turns may be equal—are inserted so as to oppose each other, this field-intensity may be directly measured. In this way the correction for the thickness of the wire is also obtained. See also Ewing, *Magnetic Induction*, &c. chapter vii.

§ 218. **Yoke Method.**—It is scarcely possible in practice that the ferromagnetic substance examined shall always have a specially prescribed form. Several methods have therefore been devised to use the material, which commercially is usually supplied in the shape of bars of various section or as wire, with the ballistic method without much trouble. J. Hopkinson [1] first endeavoured to diminish the considerable self-demagnetisation of a bar-shaped specimen by enclosing it in a simple or double *closed yoke* (fig. 85), AA, of large cross-section, and of the softest iron. The magnetisation curve is now much less sheared to the right than would be the case without such a closed yoke (§§ 17, 206). The directrix for a given closed yoke may be determined by means of a normal specimen, the magnetisation curve of which is known.

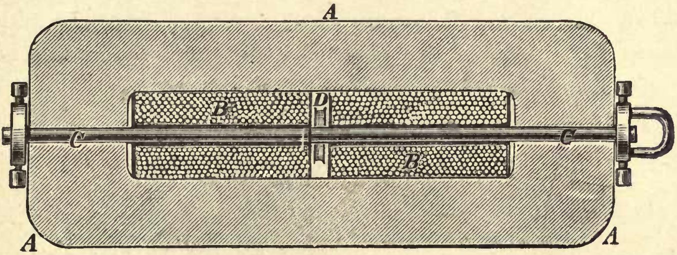

FIG. 85

The induction in the test-piece within the yoke was determined ballistically by J. Hopkinson. In his original experiments the specimen bar consisted of two parts, CC (fig. 85), separated in the middle. By means of a handle the part on the right could be suddenly pulled out so far that, by means of a spring, the secondary coil D sprang out of the field between the primary coils BB. The momentary current then measures the flux of induction which had existed in the place in question. The surface of separation between the two halves is obviously in an unfavourable place, where the irregularities due to it (§ 151) directly influence the secondary coil. The two halves may, of course, be left in position, and, the secondary coil being fixed, step-by-step measurements may be made in the usual manner. Hopkinson's arrangement does good service, especially if a

[1] J. Hopkinson, *Phil. Trans.* vol. **176**, p. 455, 1885.

suitable shearing of the induction curve is not omitted, and the specimen bar is not of highly permeable soft iron like the yoke; for the magnetic reluctance of a 'harder' kind of iron or steel is, of course, far greater in comparison with that of the yoke.

§ 219. **Various Forms of Closed Yoke.**—The arrangement is more favourable when, as proposed by Ewing,[1] two test bars are

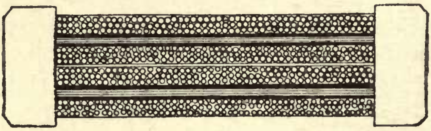

FIG. 86

used, and their ends connected by two massive yokes (fig. 86, p. 336). The secondary coils are best arranged about the middle of each bar. One of the yokes is so arranged that when it is pulled off, the secondary coils come away with it.

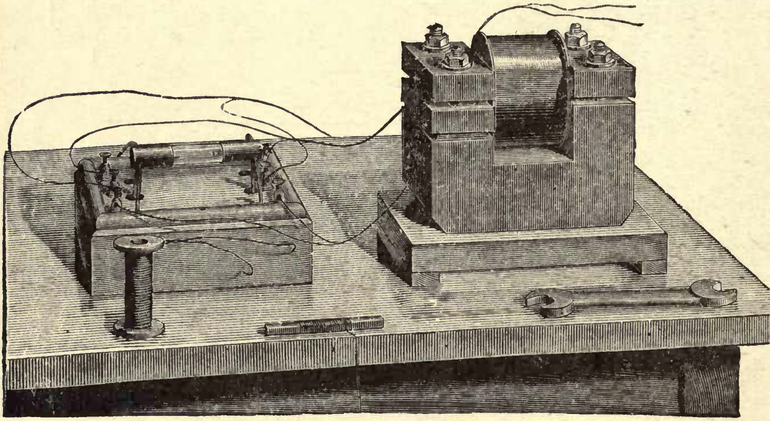

FIG. 87

In the apparatus represented in fig. 85 the test bars are inserted in the perforations in the yoke—in which case they must fit well, so as to avoid air-spaces; hence a very accurate form of section is necessary. In this respect, the simple connection at the ends, as in fig. 86, is more practical. The rods, strips, or

[1] Ewing, *Magnetic Induction*, §§ 60, 161.

bundles of wire, of any cross-section, need only be of a definite length, and with their ends cut off straight and smooth. The magnetic pull itself produces a close contact, which can be still further improved by external pressure—as by screws, for example (§ 152).

An apparatus for practical purposes depending on the method in question is that described by Corsepius[1] under the name of *Siderognost* (σιδηρός, iron), in which a simple U-shaped yoke is used. Fig. 87 represents the smaller form of this apparatus, which, after what has been stated, needs no further explanation.

A simple arrangement has recently been described by Behn-Eschenburg,[2] by means of which rapid comparative measurements of various kinds of iron may be made (fig. 88). The

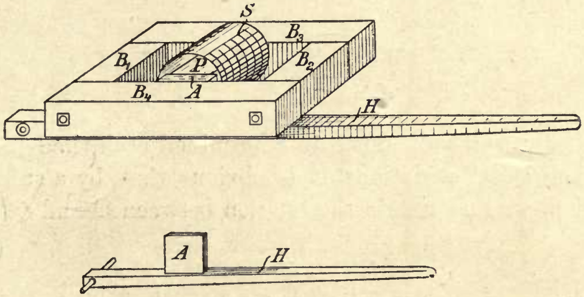

FIG. 88

test-piece P is fixed to the posterior bar of a closed yoke $\overline{B_1 B_3 B_2 B_4}$; between P and the front bar is an iron plate A carrying the secondary coil A, which, by means of a lever H, can be suddenly withdrawn, as in Hopkinson's arrangement.

§ 220. **Case of Great Self-induction.**—It was stated in § 216 that the ballistic method, in its ordinary form, fails when the self-induction is too high. This defect may be partially remedied by bringing into the circuit of the magnetising coil a considerable non-inductive resistance, by which variations of current, and therewith magnetic variations, are less re-

[1] Corsepius, *Untersuchungen zur Konstruktion magnetischer Maschinen*, pp. 46–61, Berlin, 1891.

[2] Behn-Eschenburg, *Elektrotechn. Zeitschr.* vol. **14**, p. 330, 1893.

tarded.[1] A corresponding higher electromotive force is then, of course, necessary.

On the other hand, this very fact may be taken advantage of. By determining the self-inductance, by any suitable method, certain conclusions can be drawn as to the properties of the ferromagnetic bodies. For, from § 153, equation (8), for a closed magnetic circuit we have, in our usual notation, as introduced there,

$$(13) \quad . \quad . \quad . \quad \frac{d\mathfrak{B}}{d\mathfrak{H}} = \frac{L}{4\pi n^2 S} A$$

Let a variation $\delta\mathfrak{H}$ of the intensity, and $\delta\mathfrak{B}$ of the induction, correspond to a small finite variation δI of the current, during which the self-inductance may be regarded as constant. Then, from the above equation,

$$(14) \quad . \quad . \quad \delta\mathfrak{B} = \frac{AL}{4\pi n^2 S}\delta\mathfrak{H} = \frac{A}{nS}\delta I$$

If the self-inductance is determined each time for a series of such small variations, it is obvious that, by a suitable integration, we may obtain the relation between $\mathfrak{B}$ and $\mathfrak{H}$. For

$$(15) \quad . \quad \mathfrak{B} = \frac{L}{4\pi n^2 S}\int_0^{\mathfrak{H}} A\,d\mathfrak{H} = \frac{1}{nS}\int_0^{I} A\,dI$$

Swinburne and Bourne [2] have described a practical method, which may be regarded as coming under this head, in which the material to be investigated is in the form of wire. They wind the wire in a mould of known dimensions, so as to form a ring, which is then wound uniformly with a primary and secondary coil. In order to dispense with the ballistic measurement of the current in the secondary, it is compensated by an equal and opposite one, so that the galvanometer is only used as a

[1] Compare p. 265, note 1. See also J. Hopkinson, *Original Papers on Dynamic Machinery*, p. 198, New York, 1893.

[2] Swinburne and Bourne, *Phil. Mag.* [5], vol. **24**, p. 85, 1887; *The Electrician*, vol. **25**, p. 648, 1890.

zero-indicating instrument. For this purpose the primary coil of the ring is connected up in the same circuit with that of an 'induction-box.' [1] The secondary coils of the latter are, on the other hand, in the same circuit with that of the ring, but in such a manner that the two induced impulses act in opposition to each other. Secondary coils are now inserted in the induction-box until compensation is obtained. Its mutual inductance is thus equal to that of the two coils of the ring. This mutual inductance, multiplied by the ratio of the number of turns [§ 176, equation (39)], gives the unknown self-inductance of the primary coil. As regards the details of Swinburne and Bourne's method, we may refer to the paper quoted.

§ 221. **Methods of J. and B. Hopkinson and of T. Gray.**— The following method has been applied by Drs. J. and B. Hopkinson [2] for determining the hysteresis with rapidly successive cyclical processes. Iron or steel wire of $\frac{1}{4}$ mm. diameter was used, and wound into a ring; on this wire n ($= 200$) turns of copper wire were coiled. Through this 'induction coil' (as explained in § 153) an alternating current was passed. The current I, as well as the impressed electromotive force E_e, were, by means of a rotating contact-maker and an electrometer provided with a condenser, determined as periodic functions of the time, and represented graphically by an (I, T)-curve and an (E_e, T)-curve. By multiplying the ordinates of the former with the resistance of the coil, and subtracting from the ordinates of the latter, a third curve is obtained, which, by equation (7), § 153,

$$(16) \qquad . \qquad . \qquad \frac{d\,(n\,\mathfrak{S})}{d\,T} = E_e - IR$$

represented this differential quotient as a function of time. Integration gives, in the first place $(n\,\mathfrak{S})$, and therefore, also, $\mathfrak{B} = (n\mathfrak{S})/nS$ as a function of T; and then, by reference to the corresponding ordinates of the (I, T)-curve, as a function of I or of $\mathfrak{H} = 4\pi n\,I/L$; that is, the unknown curve of induction is

[1] Such an induction box is the analogue of a resistance box. It contains two sets of coils, whose mutual inductance may, within certain limits, be regulated, either step by step or continuously. See Graetz, *Wied. Ann.* vol. **50**, p. 769, 1893.

[2] J. and B. Hopkinson, *The Electrician*, vol. **29**, p. 510, 1892.

ultimately obtained. Its area in the usual manner represents the dissipation of energy due to hysteresis.

Besides from periodic current curves, either by the above method, or by an inversion of Sumpner's method (§ 156), the curve of induction may, as T. Gray [1] has done, be also deduced from the curve representing the rise of the current on making. As this method, which is of theoretical interest, has been discussed and graphically explained, we shall be content with a few statements in reference to what is rather a circuitous method of experimenting.

Measurements and graphs of the time-course of rapidly varying currents have been obtained by Blaserna [2] with rotating contact arrangements, and by Helmholtz by means of his well-known contact-pendulum.[3] In subsequent investigations these methods, more or less modified, have been used. For determining his current curves T. Gray at first used a method like the one given by Hopkinson. With the very high time ratio of his electromagnet (compare fig. 41, p. 243) the slow changes of current could be followed with a rapidly vibrating, but strongly damped, reflecting galvanometer. The mirror reflected a spot of light on a chronograph drum rotating about a horizontal axis, and covered with co-ordinate paper. By following the motion of the spot of light with a pencil, curves were obtained of which we have represented in fig. 40 (p. 242) and 41 (p. 243) some observed with a *closed* magnetic circuit, together with the induction-curves deduced from them in the manner there described.

D. *Magneto-optic Methods*

§ 222. **Kerr's Phenomenon.**—The intensity of a field may be measured optically (§ 199); in like manner we may determine magnetisation by means of Kerr's magneto-optic phenomenon (§ 10). Linearly polarised monochromatic light is allowed to fall (normally in preference) on the reflecting magnet, and the reflected light is then investigated by means of a special

[1] T. Gray, *Phil. Trans.* vol. **184** A, p. 531, 1893. Compare also Evershed's criticisms, *The Electrician*, vol. **32**, p. 316, 1894.

[2] Blaserna, *Giornale di Scienze Naturali ed Economiche*, vol. **6**, p. 38, 1870.

[3] Helmholtz, *Wissensch. Abhandl.* vol. **1**, p. 629, Leipzig, 1882.

polariser and analyser. A simple rotation ε of the plane of polarisation, without any disturbing ellipticity, is then observed which is proportional to the normal component of magnetisation, as has been mentioned in introducing that vector (§ 11); hence we put

$$\varepsilon = K\mathfrak{J}_\nu \tag{17}$$

K is a negative or positive factor of proportionality, and is known as *Kerr's constant*; for a given metal it depends on the wave-length, but scarcely at all on the temperature; the absolute values are known for the ferromagnetic metals.

Fig. 89 represents an experimental arrangement used by the author.[1] The reflecting polished plate M to be investigated is fixed to one massive pole-piece, P, of a powerful electro-

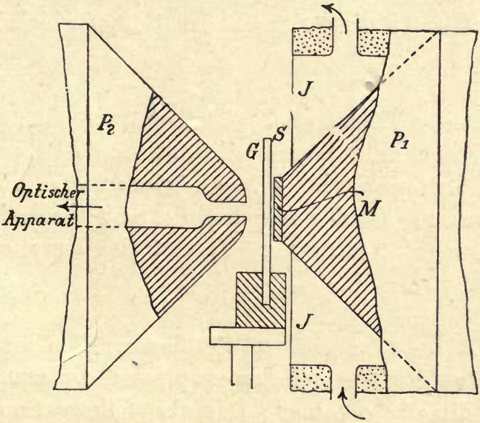

FIG. 89

magnet; the other pole-piece is bored so as to allow incident and transmitted light to pass. In addition to the magnetisation to be deduced from equation (17), the corresponding abscissa of the curve of magnetisation has also to be determined. For

[1] Du Bois, *Phil. Mag.* [5], vol. **29**, p. 293, 1890; *Wied. Beibl.* vol. **14**, p. 1156, 1890. Compare also *Wied. Ann.* vol. **39**, p. 25, 1890; and vol. **46**, p. 545, fig. 1, 1892. Since, from equation (17), the rotation is proportional to the normal component of magnetisation, it would theoretically be possible to determine the distribution of $\mathfrak{J}_\nu$—for instance, by means of a very thin 'proof mirror' (see § 208).

this purpose the field intensity $\mathfrak{H}_t$, normal to the mirror and immediately in front of it, is measured, by means of standard glass plate G, silvered at the back S (compare § 200). On account of the normal continuity of the total induction the value of this vector within the plate, that is $\mathfrak{B}_t'$, is equal to that of $\mathfrak{H}_t$ (compare note, p. 83). We obtain in this manner

$$(18) \quad . \quad \mathfrak{I} = \text{funct.}\,(\mathfrak{B}_t') = \text{funct.}\,(\mathfrak{B}_t' + 4\,\pi\,\mathfrak{I})$$

that is, a direct relation between the two vectors $\mathfrak{I}$ and $\mathfrak{B}_t'$. Since, however, it is not usual to represent this relation graphically, but ordinarily to introduce the single independent variable $\mathfrak{H}_t$ as abscissa, the curve of magnetisation, that is,

$$(19) \quad . \quad . \quad . \quad \mathfrak{I} = \text{funct.}\,(\mathfrak{H}_t')$$

may be conveniently obtained by shearing backward from the axis of ordinates to a directrix, the equation of which is

$$(20) \quad . \quad . \quad . \quad \mathfrak{H} = -\,4\,\pi\,\mathfrak{I}$$

§ 223. **Kundt's Phenomenon.**—The rotation of the plane of polarisation when light passes at right angles through thin transparent ferromagnetic layers, transversely magnetised, which is known as *Kundt's phenomenon*, may theoretically also be used for measuring purposes. The rotation ε here observed is

$$(21) \quad . \quad . \quad . \quad \varepsilon = \Psi\,\mathfrak{I}\,d$$

in which d is the thickness of the layer, Ψ a factor called *Kundt's constant.*[1] The relation expressed by equation (18) is thus also directly obtained; this also follows from the fact that the curve of magnetisation of a plate is the limiting curve, which corresponds to the greatest possible demagnetising factor $N = 4\,\pi$ (§§ 30, 33). These magneto-optical methods therefore are suited only for investigating the saturation-stage (§ 13) of magnetisation; for in the lower stages the curve of magnetisation differs so little from the straight line symmetrical with the directrix [equation (20)], the equation of which is

$$(22) \quad . \quad . \quad . \quad \mathfrak{H} = +\,4\,\pi\,\mathfrak{I}$$

[1] Kundt, *Wied. Ann.* vol. **23**, p. 228, 1884; and vol. **27**, p. 191, 1886. Compare also du Bois, *Wied. Ann.* vol. **31**, p. 966, 1887.

that the differences are quite within the range of errors of observation. This case is the most striking example of the manner in which all special properties of the ferromagnetic substance are concealed by its shape.

Although the rotation with transmitted light may in certain circumstances be almost ten times that on reflection, it cannot for various reasons be so accurately determined. In the former method we are restricted to thin electrolytic layers of metal, and cannot investigate any given mass of material; in consequence of this the method is in general less to be recommended than that depending on Kerr's phenomenon.

E. *Hall's Phenomenon. Bismuth Spiral*

§ 224. **Hall's Phenomenon. Bismuth Spiral.**—It has recently been shown by Kundt[1] that the angle of deflection β of the equipotential lines (compare fig. 69, p. 310), in thin ferromagnetic layers transversely magnetised, is proportional to the rotation ε of the plane of polarisation of the transmitted light. He concludes from this, that that angle, or, in other words, the potential of difference between the two 'Hall's electrodes,' must also from (§ 223) be proportional to the magnetisation, and hence be suited for measuring it. This mode of measurement, which is perhaps capable of development, has been just as little used for determining field-intensity as the corresponding phenomenon in non-magnetic metals (§ 201). The method might specially be suited for investigating the stage of saturation, for we are dealing with the limiting curve corresponding to the value $N = 4\pi$ discussed in the previous paragraph. It is to be observed that the measurement of the difference of potentials of the Hall electrodes is far more accurate and convenient than that of the rotation of the plane of polarisation on transmission.

§ 225. **Bruger's Apparatus.**—The method of measuring a field by means of a bismuth spiral, described in § 202, has been used by Bruger as the basis of a magnetic apparatus, the most recent form of which is represented in fig. 90.[2] A handle-shaped yoke of circular section encloses the test-bar, which ought to

[1] Kundt, *Wied. Ann.* vol. **49**, p. 257, 1893.

[2] The original form of the apparatus was shown at the Frankfort Electrotechnical Congress in 1891. (See *Berichte der Sektions-Sitz*, p. 87, Frankfort, 1892.)

have the same section as the yoke. An air-gap is left (on the right in the figure), within which the field can be measured by means of a bismuth spiral. The width of the slit is determined each time by the micrometric arrangement represented. The

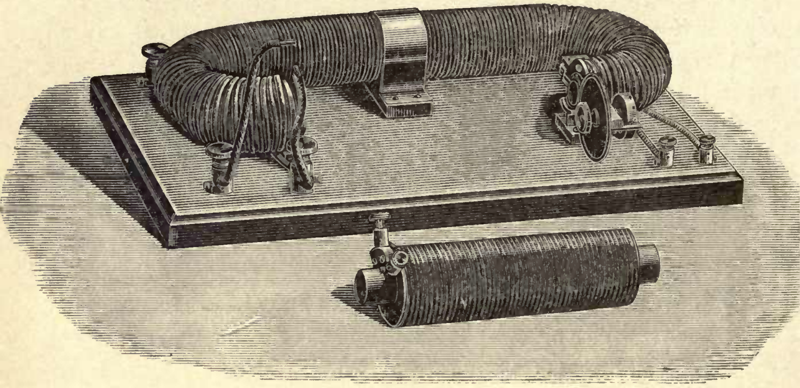

FIG. 90.—$\frac{1}{5}$ nat. size

corresponding magnetomotive force, expressed direct in ampere-turns, is denoted by M_e' (p. 196, note 1):

(23) . . . $M_e' = 0{\cdot}8\,\mathfrak{H}\,d$

Let the total value of the ampere-turns be M_t'; then for the yoke there remains M_j' and for the test-bar M_p'; so that

(24) . . $M_p' + M_j' = M_t' - M'$

If S is the constant cross-section, then, independently of leakage, the flux of induction is $\mathfrak{G} = \mathfrak{H}\,S$. Hence, we can plot M_p' as a function of $\mathfrak{G}$, which is the object of the apparatus. Before doing this, M_j' must be determined once for all as a function of $\mathfrak{G}$ by means of a standard test-bar of the same kind of iron as the yoke. Leakage may be allowed for by taking the bismuth spiral of somewhat larger diameter than the slit. Compare what has been said on the behaviour of bismuth spirals in § 202.

F. *Traction Methods*

§ 226. **Thompson's Permeameter.**—We mentioned in § 107 the experiments of Shelford Bidwell on toroids divided diametrically,

as well as with divided bars. This method of using the tractive force to measure the induction or the magnetisation, forms the basis of a class of apparatus to which reference has been made.

Silvanus Thompson has developed the latter arrangement mentioned, so as to form an apparatus for practical purposes, which he has called a *permeameter*.[1] This is represented in fig. 91. The specimen to be tested passes through a hole in the top of the yoke, fitting very closely, and through a coil, so that its lower end, faced true, rests on a portion of the yoke, which is

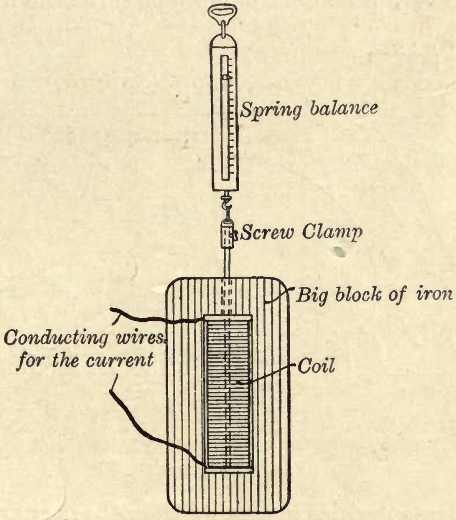

FIG. 91

also faced true. The force required to pull off the bar is determined by a spring balance. As the coil remains at rest when the bar is detached, Thompson takes into account only the first member of equation (13), § 104 and therefore puts

$$\mathfrak{F} = \mathfrak{Z} S = 2\pi \mathfrak{J}^2 S \qquad (25)$$

in which S is the section of the bar, $\mathfrak{F}$ the total tractive force. If the latter is expressed in kilogrammes-weight, and the above quantities are distinguished, as in § 103, by being written $\mathfrak{F}_1$

[1] Silv. Thompson, *Dynamo-electric Machinery*, 4th edition, p. 138, 1892.

and $\mathfrak{Z}_1$, we obtain, with sufficient approximation, the following formula:

$$(26) \qquad \mathfrak{J} = 395 \sqrt{\mathfrak{Z}_1} = \frac{395}{\sqrt{S}} \sqrt{\mathfrak{F}_1}$$

The spring balance may, of course, be so graduated that the value of $\mathfrak{J}$ may be at once read off.

All the sources of error (§ 107) which we have already discussed in detail when speaking of experiments on lifting power, of course also influence measurements with the permeameter. The joint is, moreover, in an unfavourable place, as the lines of induc-

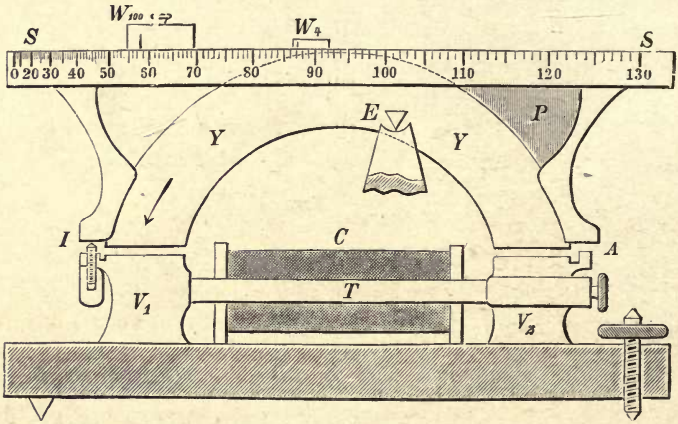

FIG. 92.—¼ scale

tion present irregularities in passing from the narrow bar into the yoke. Ewing has, accordingly, proposed that the joint should be in the middle of the bar.

§ 227. **Magnetic Balance.**—In order to get rid of these drawbacks, the author has designed a *magnetic balance*, in which there is no actual separation of two ferromagnetic substances in contact. An outline section of the apparatus is given in fig. 92, while fig. 93 gives a perspective view. The test-bar T is clamped automatically (§ 219) between the wrought-iron cheeks V_1 and V_2, after having been cut exactly to 15 cm., and, if necessary, turned to a diameter of 1·128 cm. This represents a cross-section of 1,000 sq. cm., which need not be strictly adhered to, but which is so far convenient that it dispenses with

[1] See G. Kapp, the *Electrician*, vol. **32**, p. 498, 1894.

all calculation. The coil C furnishes fields up to 300 C.G.S. units. It is 4π cm. in length,[1] and has 100 turns, so that the field $\mathfrak{H}$ inside it is at once obtained if the current (in amperes) is multiplied by 10. Above the cheeks, and at a slight distance, is a yoke $\overline{YY}$, which also forms the beam of the balance. Its knife-edge rests excentrically on the apparatus. Equilibrium is restored by the lead block P.

The magnetic attractions on each side, which are equal by reason of symmetry, produce, nevertheless, unequal couples, owing to the unequal leverage. The resultant, as was shown by

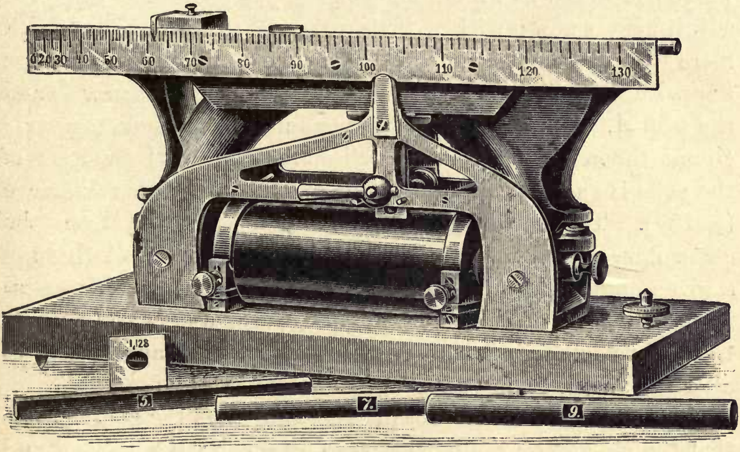

FIG. 93.—$\frac{1}{4}$ scale

special ballistic experiments, is proportional, within the range of the experiments $(0 < \mathfrak{B} < 16{,}000)$, to the square of the flux of induction $\mathfrak{B}$ in the middle of the test-bar. This shows that with the corresponding comparatively weak magnetisation of the cheeks and of the yoke the leakage is unaltered (compare §§ 109, 110). On account of the comparatively low fields produced, $\mathfrak{J}$ and $\mathfrak{B}$ may, further, be taken as proportional (compare §§ 11, 59). The yoke is attracted downwards on the left, and this action is compensated by sliding weights W_{100} or W_4. The instrument is so adjusted that, by multiplying the reading of the scale by 10 $(=\sqrt{100})$ or 2 $(=\sqrt{4})$ respectively, the

[1] It is, in fact, somewhat shorter (12·2 cm.), so as thereby to compensate in a simple manner for the influence of the ends.

absolute value of the magnetisation is obtained, if the section of the test-piece is 1 sq. cm.; otherwise, after plotting the curves, the ordinates must be divided by the cross-section. It is immaterial, in this respect, whether the section is circular, square, or of any other form, or whether the specimen consists of a bundle of wires, ribbons, or strips; it should, if possible, not differ by more than 10 per cent. from 1 sq. cm.

Since the magnetic equilibrium is by the nature of the case always unstable, no fixed position can be read off as on the pointers of a balance; but by trial a position of the sliding weight is found at which the yoke is just detached from the adjustment screw; with a little practice this is effected with more than sufficient accuracy. The yoke oscillates like a Morse key with very little play (0·1—0·2 mm.) over the screw I, and the stop A. By its tilting the reluctance of the whole magnetic circuit theoretically alters to a slight extent, as the motion of the yoke is subject to the principle of least magnetic reluctance (§ 158). The necessity of this follows, moreover, from the elementary consideration, that when the yoke is tilted through a given angle, the width of the left air-gap decreases [increases] more than that of the right increases [decreases]; these variations of reluctance are, however, so small as not appreciably to alter the value of the flux of induction.

The tilting, to the left for instance, produces the consequence that the air-gap on the left becomes narrower; the leakage, and therefore the efficient cross-section for the transmission of the lines of induction, is less, and accordingly, with a constant flux of induction, the total tractive force becomes greater there (§ 109); with the air-gap on the right the reverse is the case. This explains in the first case the unstable equilibrium, and secondly the necessity that the detachment must take place with a definite invariable width of both air-gaps. The latter amount to about 0·3 mm.; the instrument is so designed that variations in this are excluded as much as possible. In graduating it by means of a standard bar,[1] the adjusting screw I on the left is so adjusted

[1] [This is the method of graduation as carried out by the Physikalisch-technische Reichsanstalt; the standard bar was once for all compared with an ovoid of the same material; see *Zeitschrift für Instrumenten-Kunde*, vol. **15**, p. 330, 1895.—H. du Bois.]

that the reading gives the correct values of magnetisation, and is then provided with a sealed cap.

§ 228. **Use of the Balance.**—In order to demagnetise all parts of the magnetic circuit as completely as possible, the yoke may be slowly raised by a lever arrangement, while the commutator in the circuit is rapidly worked (§ 207); when the instrument is not in use the yoke is stopped at the highest position. If this method is not sufficient to destroy the effects of the previous magnetic history of the yoke, which is of great importance, a special demagnetising coil must be used. The demagnetisation

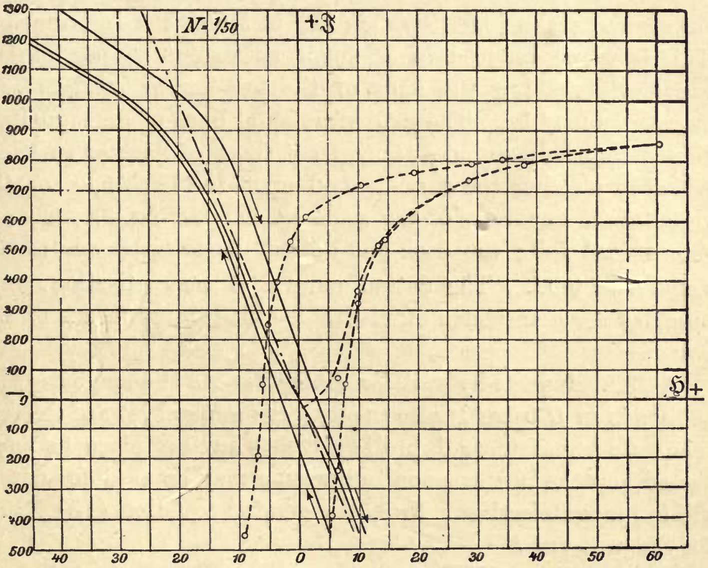

FIG. 94

of the specimen may, from § 207, be effected by pushing the coil towards the observer, and drawing the specimen slowly out, while the reversals are being continued.

According to Table I, p. 41, the demagnetising factor $\overline{N} = 0{\cdot}12$ responds to a dimensional ratio $m = 15$; in the magnetic balance $\overline{N}$ is about equal to $0{\cdot}02$, corresponding to the ratio 45; this is therefore apparently trebled by applying the yoke to close the circuit. The same value of $\overline{N}$ would be

obtained by bending the 15 cm. bar into a toroid, until its ends formed a slit $\frac{1}{4}$ mm. in width. $\overline{N}$ is moreover not constant; in plotting the curves it is therefore better, instead of starting from straight directrices (§ 17), to plot the curves from suitable lines of demagnetisation (fig. 21, p. 131). In graduating the apparatus three of these are determined once for all, one for magnetisation increasing from zero, and two for the ascending or descending branches of hysteristic loops of cyclic magnetising processes. By their use we automatically correct the influence of the magnetic reluctance of the cheeks V_1 and V_2, of the yoke $\overline{Y\,Y}$, of the air-gaps with their leakage, as well as of the small resistance on the joint between the test and the instruments. This is to be reduced to as small an extent as possible by previously making the ends of the test-bar as straight and smooth as may be. These 'instrumental lines of demagnetisation' may be drawn on a transparent sheet of horn or gelatine, which in plotting the curve is laid upon the left upper or the right lower quadrant of the scale paper. In fig. 94 they are represented full; the dash-and-dotted line corresponds to the value $\overline{N} = 0{\cdot}02$. The dotted curves of magnetisation were obtained for a specimen of malleable cast iron by means of the balance.[1]

§ 229. **Magneto-hydrostatic Methods.**—As the magnetic rise of liquids in U-tubes is also ruled by the general laws of electromagnetic stress (note 3, p. 314), it may in this place be mentioned how the phenomenon in question may be used for determining magnetisation. From the general equation (15) § 203, for the pressure P

$$P = \int_0^{\mathfrak{H}} \mathfrak{J}\, d\,\mathfrak{H}$$

we obtain by differentiation

$$\mathfrak{J} = \frac{\partial P}{\partial \mathfrak{H}} \qquad (27)$$

[1] For further details as to the construction and use of the instruments reference must be made to the following publications: du Bois, *Ber. Sekt. Sitz. Elektrotechn. Kong.*, Frankfort, 1891, p. 77; *Elektrotechn. Zeitschrift*, vol. **13**, p. 579, 1892; *Zeitschrift für Instrumenten-Kunde*, vol. **12**, p. 404 1892.

If, therefore, the pressure P is determined as a function of $\mathfrak{H}$, the magnetisation may be obtained by graphical differentiation, in which of course the magnetic intensity is now assumed to be known. For weak para- or diamagnetic liquids of constant susceptibility this process reduces to a very simple method of determining that number by the equation (16) § 203, which applies to this case, into which, however, we need not enter. As regards liquid ferromagnetic bodies, we need only consider amalgams of iron, cobalt, and nickel. Quincke has made measurements with the former which have been discussed by the author with the aid of the above equation.[1] This method has not, hitherto, been any further developed.

[1] Quincke, *Wied. Ann.* vol. **24**, p. 374, 1888; du Bois, *Wied. Ann.* vol. **35**, p. 156, 1888. The magnetic behaviour of the amalgams in question can scarcely be considered as sufficiently investigated.

INDEX OF NAMES

The references are to the Articles

INDEX OF SUBJECTS

The references are to the Articles

NOMENCLATURE[1]

	L.	M.	T.	Article
A, Power	2	1	—3	148
D, Distance	1	0	0	22
D, Density	—3	1	0	203
d, Thickness or width	1	0	0	75
E, Electromotive force	$\frac{3}{2}$	$\frac{1}{2}$	—2	64
F_H, Hopkinson's function	—	—	—	96
I, Electrical current (in decamperes)	$\frac{1}{2}$	$\frac{1}{2}$	—1	5
I', Electrical current (in amperes)	—	—	—	83
I_e, Steady current	$\frac{1}{2}$	$\frac{1}{2}$	—1	153
J, Impedance	1	0	—1	155
K, Moment of inertia	2	1	0	23
K, Kerr's constant	$\frac{1}{2}$	$-\frac{1}{2}$	1	222
L, Length	1	0	0	5
M, Magneto-motive force	$\frac{1}{2}$	$\frac{1}{2}$	—1	119
N, Demagnetising factor	0	0	0	24
N, Frequency	0	0	—1	155
n, Number of windings	0	0	0	5
P, Hydrostatic pressure	—1	1	—2	194
Q, Quantity of electricity	$\frac{1}{2}$	$\frac{1}{2}$	0	2
R, Ohmic resistance	1	0	—1	2
r, Radius	1	0	0	5
S, Area, Section	2	0	0	2
T, Time	0	0	1	64
V, Volume	3	0	0	22
V, Permeance	1	0	0	119
X, Reluctivity	—1	0	0	119
Y, Inductive resistance	1	0	—1	155
$\mathfrak{A}$, Auxiliary vectors	$\frac{1}{2}$	$\frac{1}{2}$	—1	44

[1] The nomenclature and notation of conceptions of frequent occurrence are here arranged in order, and they are not therefore entered in the Index of Subjects. The dimension refers to the absolute system of units (§ 5); the number of the article is that in which the special term first occurs or is defined. In many cases, $\mathfrak{l}$, $\mathfrak{m}$, $\mathfrak{n}$, denote direction-cosines (§ 35); x, y, z, ν, τ, suffixes indicating direction (§§ 34, 53); e, i, t, suffixes indicating the source. With reference to the accents on letters, see §§ 51, 97. Bars over letters stand for mean values (§§ 24, 75).

	L.	M.	T.	Article
𝔅, Magnetic induction	$-\frac{1}{2}$	$\frac{1}{2}$	-1	10
ℭ, Current flow	$-\frac{3}{2}$	$\frac{1}{2}$	-1	44
𝔢, Excentricity	0	0	0	29
𝔉, Mechanical force	1	1	-2	21
𝔉, Vector quantity	—	—	—	34
𝔊, Flux of induction	$\frac{3}{2}$	$\frac{1}{2}$	-1	61
ℌ, Magnetic intensity	$-\frac{1}{2}$	$\frac{1}{2}$	-1	4
$ℌ_C$, Coercive intensity	$-\frac{1}{2}$	$\frac{1}{2}$	-1	149
ℑ, Magnetisation	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	11
$ℑ_E$, Evanescent magnetisation	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	149
$ℑ_R$, Residual magnetisation	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	149
𝔎, Moment (of a torque)	2	1	-2	23
𝔱, Retentivity	0	0	0	149
𝔐, Magnetic moment	$\frac{5}{2}$	$\frac{1}{2}$	-1	6, 22
𝔪, Dimensional ratio	0	0	0	24
𝔪, Axial ratio	0	0	0	29
𝔑, Normal	1	0	0	34
𝔫, Function, sign of	0	0	0	80
𝔭, Transformation-ratio	0	0	0	181
𝔯, Convergence of the magnetisation	$-\frac{3}{2}$	$\frac{1}{2}$	-1	49
𝔲, Energy per unit volume	-1	1	-2	148
ℨ, Resultant tension	-1	1	-2	102
α, Angle	0	0	0	5
Γ, Gravitation potential	—	—	—	69
ϵ, Rotation of the plane of polarisation	0	0	0	199
θ, Time-ratio	0	0	1	154
κ, Magnetic susceptibility	0	0	0	14
Λ, Self-inductance	1	0	0	153
μ, Magnetic permeability	0	0	0	14
ν, Leakage coefficient	0	0	0	78
Ξ, Mutual inductance	1	0	0	177
ξ, Magnetic reluctivity	0	0	0	14
τ, Period	0	0	1	23
Υ, Magnetic potential	$\frac{1}{2}$	$\frac{1}{2}$	-1	45
Φ, Potential	—	—	—	39
χ, Difference of phase	0	0	0	155
Φ, Kundt's constant	$-\frac{1}{2}$	$-\frac{1}{2}$	1	223
ω, Verdet's constant	$-\frac{1}{2}$	$-\frac{1}{2}$	1	199

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